A generalization of the Conway algebra and 4-variable polynomial invariants

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Previous works

- 1984, V.F.R Jones — Jones polynomial V.
  - For the Conway triple L_+, L_-, L_0,

$$\frac{1}{t}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0).$$

  - For the trivial knot T_1,

$$V(T_1) = 1.$$

Previous works

• 1987, J. Hoste, A. Ocneanu, K. Millett, P. J. Freyd, W. B. R. Lickorish, D. N. Yetter, and J. H. Przytycki, P. Traczyk — 2-variable polynomial invariant P, which is called *HOMFLY-PT polynomial.*

- For the Conway triple L_+, L_-, L_0,

$$v^{-1}P(L_+) - vP(L_-) = wP(L_0).$$

- For the trivial knot T_1,

$$P(T_1) = 1.$$

Previous works

• 2016, M. Chlouveraki, J. Juyumaya, K. Karvounis, S. Lambropoulou and W. B. R. Lickorish [1] — 3-variable polynomial invariant Θ.

- For mixed crossings,

$$v^{-1}\Theta(L_+) - v\Theta(L_-) = (v^{\frac{1}{2}} - v^{-\frac{1}{2}})\Theta(L_0).$$

- For a link diagram L of a split union of r knots,

$$\Theta(L) = E^{1-r}P(L),$$

  where E is an indeterminate and P is the HOMFLY-PT polynomial.

Previous works

• 2017 L. H. Kauffman and S. Lambropoulou [2] — 4-variable polynomial invariant $H[R]$.

- For mixed crossings,

$$H[R](L_+) - H[R](L_-) = zH[R](L_0)$$

- For a link diagram L of a split union of r knots,

$$H[R](L) = E^{1-r}R(L),$$

where E is an indeterminate and R is the regular isotopy version of the HOMFLY-PT polynomial.

Goal

- J. H. Przytycki and P. Traczyk constructed HOMFLY-PT polynomial by using the Conway algebra.
- L. H. Kauffman and S. Lambropoulou constructed 4-variable polynomial, roughly speaking, by applying different skein relations on self crossings and mixed crossings.

Goal : Construct an algebraic structure such that

1. it contains the Conway algebra,
2. we can obtain 4-variable polynomial invariant of L.H.Kauffman and S. Lambropoulou.

Definition 2.1 ([3] J.H. Przytycki and P. Traczyk)

Let $\mathcal{A}$ be an algebra with $\circ, /$ two binary operations on $\mathcal{A}$, $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$. The quadruple $(\mathcal{A}, \circ, /, \{a_n\}_{n=1}^{\infty})$ is called *a Conway algebra* if it satisfies the following relations:

1. $(a \circ b)/b = a = (a/b) \circ b$ for $a, b \in \mathcal{A}$,
2. $a_n = a_n \circ a_{n+1}$ for $n \in \mathbb{N}$,
3. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for $a, b, c, d \in \mathcal{A}$.

Remark 2.2

The original definition of the Conway algebra [3] has three more relations

- $(a \circ b)/(c \circ d) = (a/c) \circ (b/d)$,
- $(a/b)/(c/d) = (a/c)/(b/d)$
- $a_n = a_n/a_{n+1}$.

We can show that they can be obtained from the relations $(a \circ b)/b = a = (a/b) \circ b$ and $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$.

Let $\mathcal{L}$ be the set of equivalence classes of oriented link diagrams. In [3], J. H. Przytyski and P. Traczyk constructed an invariant W, we call it *the Conway-type invariant on the Conway algebra* $(\mathcal{A}, \circ, /, \{a_n\}_{n=1}^{\infty})$ satisfying the following properties:

1. For the trivial link T_n of n components $W(T_n) = a_n$,
2. For each crossing the following relation holds :

$$W(L_+) = W(L_-) \circ W(L_0),$$

where L_+, L_-, L_0 are the oriented Conway triple, described in Fig. 1.

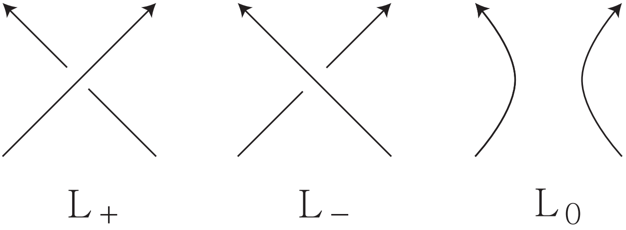


Figure: The oriented Conway triple

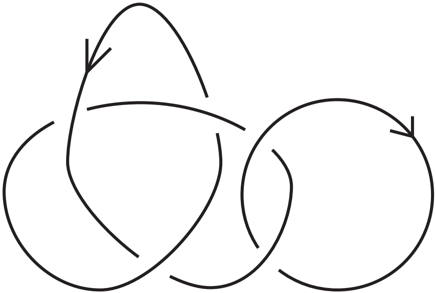
Construction

Let $L = L_1 \cup \cdots \cup L_r$ be an oriented link diagram of r components. Fix a base point b_i on each component L_i. Suppose that we walk along the diagram L_1 according to the orientation from the base point b_1 to itself, then we walk along the diagram L_2 from the base point b_2 to itself and so on. If we pass a crossing c first along the under-arc(or over-arc), we call *c a bad crossing*(or *a good crossing*, respectively). We do crossing change for all bad crossings or splice bad crossings. To specify the crossing c of L, we denote the diagram, in which the crossing c has $sgn(c) = +1$($sgn(c) = -1$) by L^c_+(L^c_-). If the crossing c is spliced, then we denote the diagram by L^c_0. Suppose that we meet the first bad crossing c with $sgn(c) = +1$. Then by applying skein relation we obtain

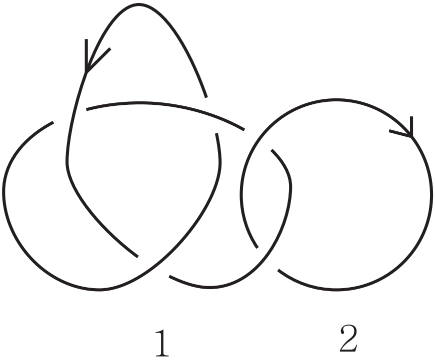
$$W(L^c_+) = W(L^c_-) \circ W(L^c_0). \tag{1}$$

Notice that if the crossing c is bad, then the number of bad crossings of L^c_- is less than the number of bad crossings of L^c_+ and the number of crossings of L^c_0 is less than the number of crossings of L^c_+. We repeat this process inductively on L^c_- and L^c_0 until we switch all bad crossings. Note that if L has no bad crossings, then the diagram L is equivalent to the trivial link diagram. For the trivial link T_n $W(T_n) = a_n$.

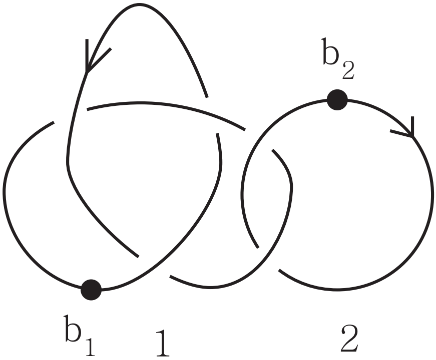
Construction of W.



Construction of W. 1. Numerate components

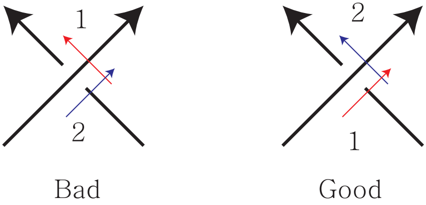


Construction of W. 2. Fix base points



Construction of W. 3. Switch bad crossings

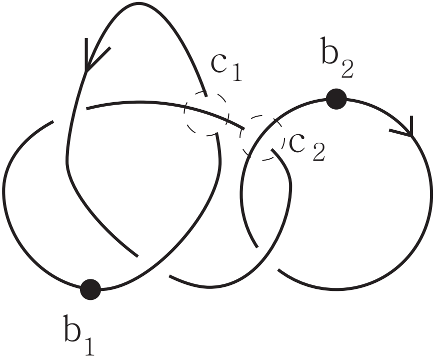
If we pass a crossing c first along the under-arc(or over-arc), we call c *a bad crossing*(or *a good crossing*).



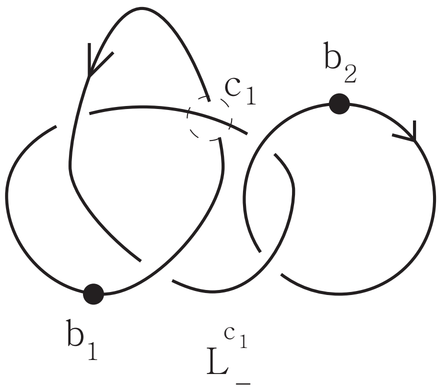
Remark 2.3

If L has no bad crossings, then the diagram L is equivalent to the trivial link diagram.

Construction of W. 3. Switch bad crossings



Construction of W. 3. Switch bad crossings



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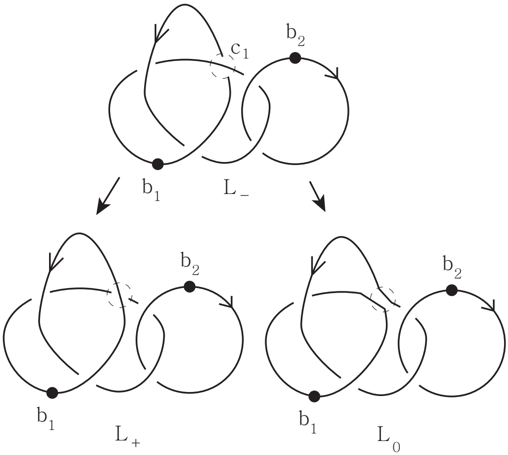

Figure: $W(L_-) = W(L_+)/W(L_0)$

Construction of W. 3. Switch bad crossings

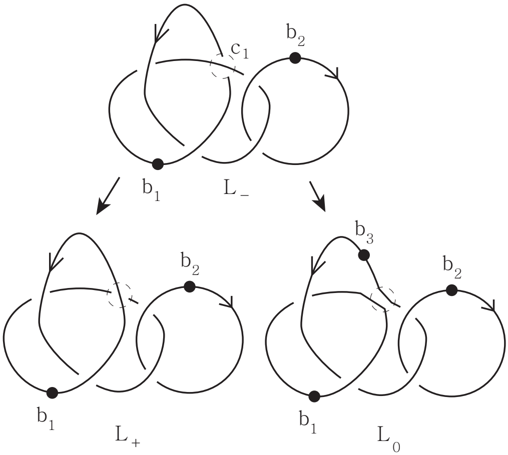

Figure: $W(L_-) = W(L_+)/W(L_0)$

Construction of W. 4. $W(T_n) = a_n$

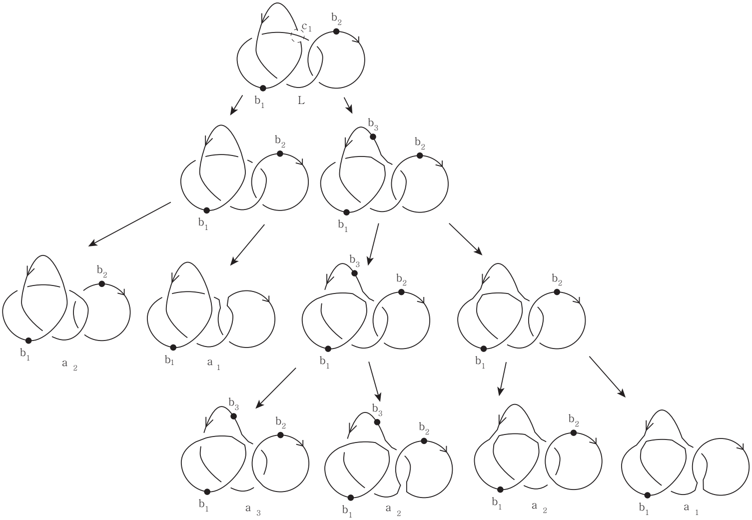

Figure: $W(L) = (a_2 \circ a_1)/((a_3 \circ a_2)/(a_2 \circ a_1))$.

Theorem 2.4 ([3])

The mapping $W : \mathcal{L} \to \mathcal{A}$ is a well-defined invariant for oriented links. That is, the value of $W(L)$ does not depend on the choice of base points and the order of links, and it is invariant under Reidemeister moves.

Proof.

Let $\mathcal{L}_k$ be the set of all ordered colored oriented link diagrams such that diagrams in $\mathcal{L}_k$ have crossings less than or equal to k. We will show that $W(L)$ is an invariant by the following steps: for every $k = 0, 1, \cdots$, on $\mathcal{L}_k$,

1. the mapping $W_b(L)$ is well-defined,
2. the value of $W_b(L)$ does not depend on the choice of base points,
3. the value of $W_b(L)$ is invariant under Reidemeister moves, which do not make the number of crossings more than k,
4. the value of $W_b(L)$ does not depend on the order of components.

by the Mathematical induction on k. □

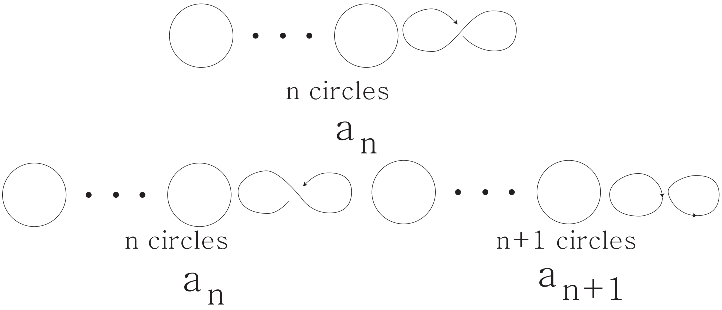

Figure: $a_n = a_n \circ a_{n+1}$.

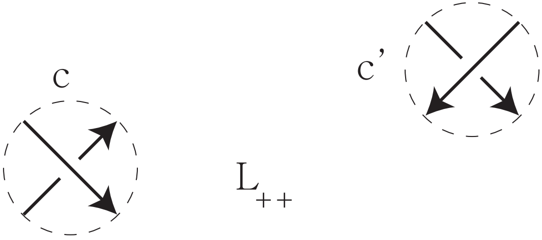

Figure: $(a \circ b) \circ (c \circ d)$.

$$\begin{aligned} W(L_{++}) &= W(L_{-+}) \circ W(L_{0+}) \\ &= (W(L_{--}) \circ W(L_{-0})) \circ (W(L_{0-}) \circ W(L_{00})) \\ &= (W(L_{--}) \circ W(L_{0-})) \circ (W(L_{-0}) \circ W(L_{00})) \\ &= W(L_{+-}) \circ W(L_{+0}). \end{aligned}$$

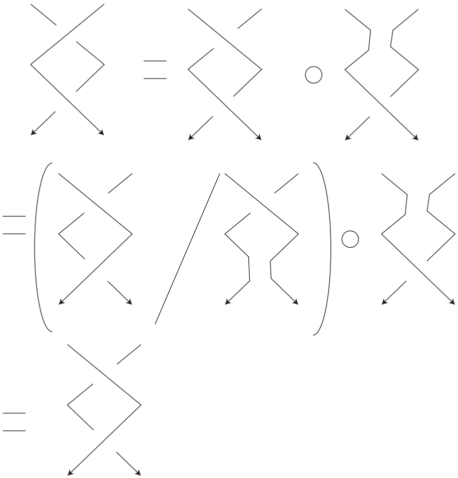

Figure: $(a \circ b)/b = a$

Example 2.5

Let $\mathcal{A} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z^{\pm 1}]$. *Define binary operations* $\circ, /$ *by*

$$a \circ b = pa + qb + z \text{ and } a/b = p^{-1}a - p^{-1}qb - p^{-1}z.$$

Let $\{a_n\}_{n=1}^{\infty}$ *be a sequence with the formula*

$$a_n = (1 - z/(1 - p - q))((1 - p)/q)^{n-1} + z/(1 - p - q).$$

Then $(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z^{\pm 1}], \circ, /, \{a_n\}_{n=1}^{\infty})$ *is a Conway algebra.*

Example 2.6

Let $\mathcal{A} = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. *Define the binary operations* $\circ$ *and* $/$ *by*

$$a \circ b = v^2 a + vzb, \ a/b = v^{-2}a - v^{-1}zb.$$

Denote $a_n = ((v^{-1} - v)/z)^{k-1}$ *for each n. This is obtained from the Conway algebra in Example 2.5 by substituting* $p = v^2, q = vz, z = 0$. *Moreover,* $W(L)$ *is the Homflypt polynomial.*

Example 2.5

Let $\mathcal{A} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z^{\pm 1}]$*. Define binary operations* $\circ, /$ *by*

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Denote $a_n = ((v^{-1} - v)/z)^{k-1}$ *for each* n*. This is obtained from the Conway algebra in Example 2.5 by substituting* $p = v^2, q = vz, z = 0$*. Moreover,* $W(L)$ *is the Homflypt polynomial.*

Definition 3.1

Let $\widetilde{\mathcal{A}}$ be a set with four binary operations $\circ, *, /, //$ on $\widetilde{\mathcal{A}}$. Let $\{a_n\}_{n=1}^{\infty} \subset \widetilde{\mathcal{A}}$. The hextuple $(\widetilde{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$ is called *a generalized Conway algebra* if it satisfies the following conditions:

- A $(a \circ b)/b = a = (a/b) \circ b$ for $a, b \in \widetilde{\mathcal{A}}$,
- B $a_n = a_n \circ a_{n+1}$ for $n = 1, 2, \cdots$,
- C $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for $a, b, c, d \in \widetilde{\mathcal{A}}$,
- D $(a * b)//b = a = (a//b) * b$ for $a, b \in \widetilde{\mathcal{A}}$,
- E $(a * b) * (c * d) = (a * c) * (b * d)$ for $a, b, c, d \in \widetilde{\mathcal{A}}$,
- F $(a * b) * (c \circ d) = (a * c) * (b \circ d)$ for $a, b, c, d \in \widetilde{\mathcal{A}}$.

Remark 3.2

Let $(\widetilde{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$ be a generalized Conway algebra. The quadraple $(\widetilde{\mathcal{A}}, \circ, /, \{a_n\}_{n=1}^{\infty})$ is a Conway algebra, and hence the Conway-type invariant can be defined on $(\widetilde{\mathcal{A}}, \circ, /, \{a_n\}_{n=1}^{\infty})$.

Definition 3.3

Let $(\mathcal{A}, \circ, /, *, //, \{a_n\}_{n=1}^{\infty})$ and $(\mathcal{A}', \circ', /', *', //', \{a'_n\}_{n=1}^{\infty})$ be generalized Conway algebras. A mapping $f : \mathcal{A} \to \mathcal{A}'$ is called *a homomorphism of generalized Conway algebras,* if $f(a \circ b) = f(a) \circ' f(b)$, $f(a * b) = f(a) *' f(b)$ and $f(a_n) = a'_n$. If f is bijective, it is called *automorphism of generalized Conway algebras.*

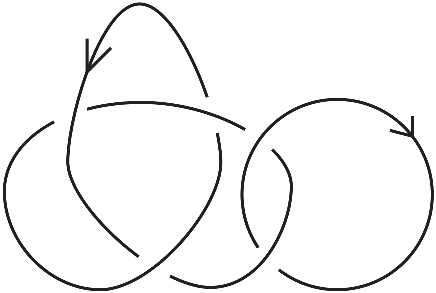
Construction of $\widetilde{W}$ valued in $\widetilde{\mathcal{A}}$

First we will define $\widetilde{W}$ for every ordered oriented link diagram. Let $L = L_1 \cup \cdots \cup L_r$ be an ordered oriented link diagram of r components. Fix a base point b_i on each component L_i. Suppose that we walk along the diagram L_1 according to the orientation from the base point b_1 to itself, then we walk along the diagram L_2 from the base point b_2 to itself and so on. If we pass a crossing c first along the under-arc(or over-arc), we call c *a bad crossing*(or *a good crossing*) with respect to the base point $b = \{b_1, \cdots, b_n\}$. We switch all bad mixed crossings. Denote the value of $\widetilde{W}$ for L corresponding to base points b by $\widetilde{W}_b(L)$. Suppose that we meet the first bad mixed crossing c with $sgn(c) = +1$. We apply the skein relation on c with the following property:

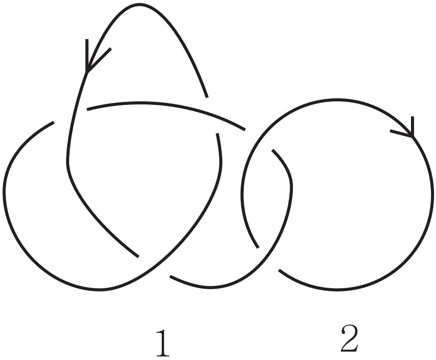
$$\widetilde{W}_b(L_+^c) = \widetilde{W}_b(L_-^c) * \widetilde{W}_b(L_0^c). \tag{2}$$

Notice that the number of bad mixed crossings of L_-^c is less than the number of bad mixed crossings L_+^c and the number of crossings of L_0^c is less than the number of crossings L_+^c. We repeat this process on L_-^c and L_0^c inductively until we switch all bad mixed crossings. If $L = L_1 \cup \cdots \cup L_r$ has no bad mixed crossings, we define $\widetilde{W}_b(L) = E^{1-r} W_b(L)$.

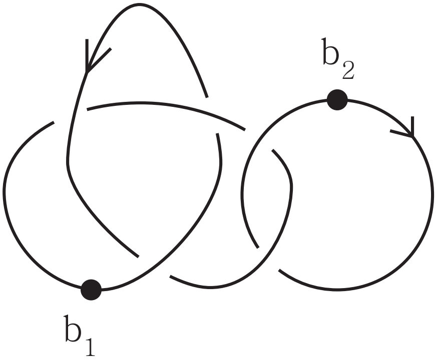
Construction of $\widetilde{W}$.



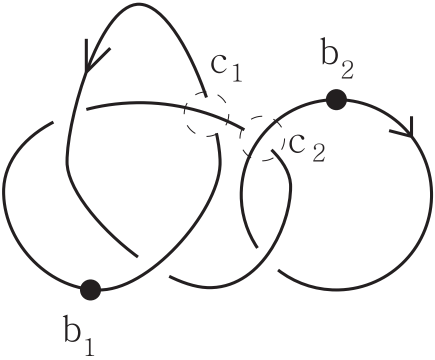
Construction of $\widetilde{W}$. 1. Numerate components



Construction of $\widetilde{W}$. 2. Fix base points



Construction of $\widetilde{W}$. 3. Switch bad mixed crossings



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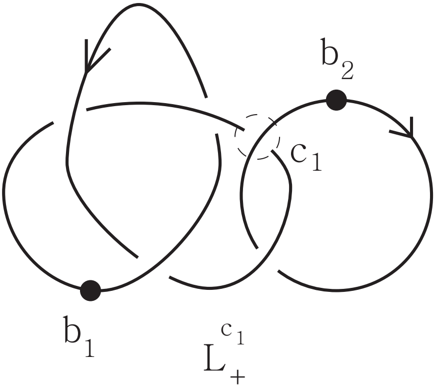

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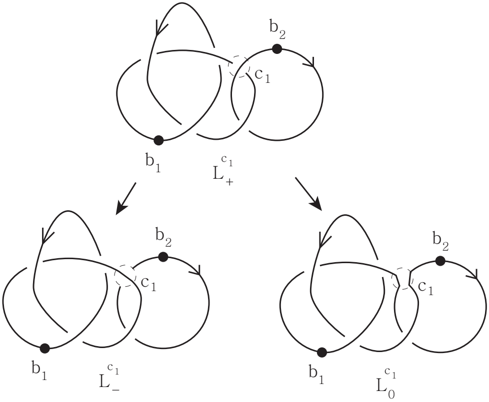

Figure: $\widetilde{W}(L_+) = \widetilde{W}(L_-) * \widetilde{W}(L_0)$

Construction of $\widetilde{W}$. 3. Switch bad mixed crossings

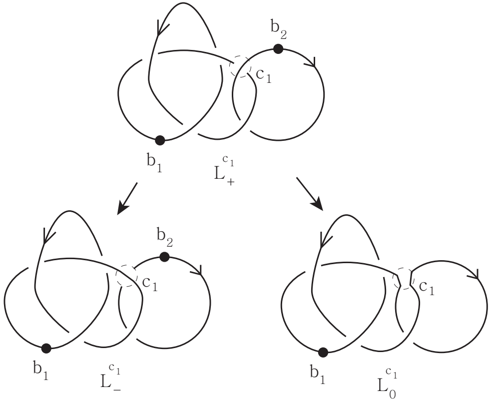

Figure: $\widetilde{W}(L_+) = \widetilde{W}(L_-) * \widetilde{W}(L_0)$

Construction of $\widetilde{W}$. 4. Values for split unions of knots

Remark 3.4

If L has no bad mixed crossings, then the diagram L is equivalent to a diagram of a split union of knots.

For a link diagram L of r components without bad mixed crossings,

$$\widetilde{W}(L) = E^{1-r} W(L),$$

where E is an automorphism of $\widetilde{A}$.

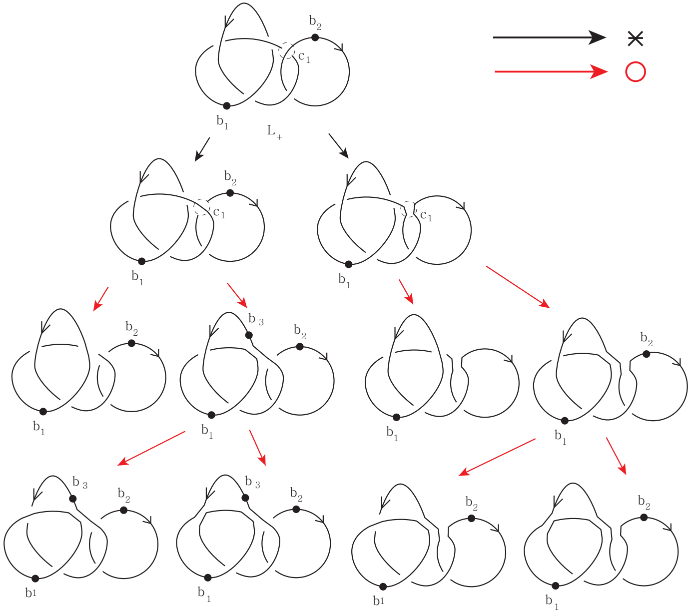

Figure: $\widetilde{W}(L) = E^{-1}(a_2/(a_3/a_2)) * (a_1/(a_2/a_1))$.

Theorem 3.5

Let $\mathcal{L}$ be the set of equivalence classes of oriented link diagrams. Let $(\widetilde{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$ be a generalized Conway algebra. Let W denote the Conway-type invariant on the Conway algebra $(\widetilde{\mathcal{A}}, \circ, /, \{a_n\}_{n=1}^{\infty})$. Let E be an automorphism on $(\widetilde{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$. Then there exists an isotopy invariant of classical oriented links $\widetilde{W} : \mathcal{L} \to \widetilde{\mathcal{A}}$ satisfying the following rules:

1. *On mixed crossings the following relation holds:*

$$\widetilde{W}(L_+^c) = \widetilde{W}(L_-^c) * \widetilde{W}(L_0^c), \tag{3}$$

2. *Let $L = L_1 \sqcup \cdots \sqcup L_r$ be a link diagram without mixed crossings. Then*

$$\widetilde{W}(L) = E^{1-r} W(L).$$

We call $\widetilde{W}$ a generalized Conway-type invariant on $(\widetilde{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$.

Proof.

Let $\mathcal{L}_k$ be the set of all ordered oriented link diagrams such that diagrams in $\mathcal{L}_k$ have crossings less than or equal to k. We will show that $\widetilde{W}(L)$ is an invariant by the following steps: for every $k = 0, 1, \cdots$, on $\mathcal{L}_k$,

1. the mapping $\widetilde{W_b}(L)$ is well-defined,
2. the value of $\widetilde{W_b}(L)$ does not depend on the choice of base points,
3. the value of $\widetilde{W_b}(L)$ is invariant under Reidemeister moves, which do not make the number of crossings more than k,
4. the value of $\widetilde{W_b}(L)$ does not depend on the order of components.

by the Mathematical induction on k. □

Theorem 4.1 ([2] L.Kauffman and S.Lambropoulou)

Let $R(w, v)$ be the regular isotopy version of the Homflypt polynomial. Then there exists a unique regular isotopy invariant of classical oriented links $H[R] : \mathcal{L} \to \mathbb{Z}[z, w, v^{\pm 1}, E^{\pm 1}]$, where z, w, v and E are indeterminates, defined by the following rules:

1. *For mixed crossings the following mixed skein relation holds:*

$$H[R](L_+) - H[R](L_-) = zH[R](L_0), \tag{4}$$

   where L_+, L_-, L_0 is an oriented Conway triple.

2. *For a split union of r knots $L := \sqcup_{i=1}^{r} L_i$, it holds that:*

$$H[R](L) = E^{1-r} R(L). \tag{5}$$

Lemma 4.2

Let $\widetilde{\mathcal{A}}$ be a commutative ring with identity. Define binary operations $\circ, *$ by

$$a \circ b = pa + qb + z,\ a/b = p'a + q'b + z',\ a * b = ra + sb + w \text{ and } a//b = r'a + s'b + w',$$

for fixed $p, q, r, s, p', q', r', s' \in \widetilde{\mathcal{A}} \backslash \{0\}$ and $z, w, z', w' \in \widetilde{\mathcal{A}}$. Fix a sequence $\{a_n\}_{n=1}^{\infty}$ valued in $\widetilde{\mathcal{A}}$. Then $(\widetilde{\mathcal{A}}, \circ, *, \{a_n\}_{n=1}^{\infty})$ is a generalized Conway algebra if and only if p, q, r, s, z, w and $\{a_n\}_{n=1}^{\infty}$ satisfy the followings:

1. *p and r are invertible and $rs = sp$,*
2. *$a/b = p^{-1}a - p^{-1}qb - p^{-1}z$ and $a//b = r^{-1}a - r^{-1}sb - r^{-1}w$.*
3. *$qa_{n+1} + z = (1-p)a_n$ for $n = 1, 2, 3, \cdots$.*

Corollary 4.3

Let $\widetilde{\mathcal{A}} = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}, w^{\pm 1}, E^{\pm 1}]$ *be an algebra. Define binary operations* $\circ, *$ *by*

$$a \circ b = v^2 a + vwb, a * b = v^2 a + vzb.$$

Put $a_n = ((v^{-1} - v)/w)^{n-1}$. *Then* $(\mathcal{A}, \circ, *, \{a_n\}_{n=1})$ *is a generalized Conway algebra. Moreover, the generalized Conway-type invariant on* $(\widetilde{\mathcal{A}}, \circ, *, \{a_n\}_{n=1})$ *is Homflypt polynomial* $v^{wr(L)} H[R](L)$ *in Kauffman-Lambropoulou version, where* $wr(L)$ *is the writhe number of* L.

Proof.

(Sketch) Let $\widehat{W}(L) = v^{-wr(L)} \widetilde{W}(L)$. We will show that $\widehat{W}$ satisfies

$$H[R](L_+) - H[R](L_-) = zH[R](L_0), \tag{6}$$

for a split union of r knots $L := \sqcup_{i=1}^{r} L_i$

$$H[R](L) = E^{1-r} R(L). \tag{7}$$

□

Theorem 4.4 ([2])

Let L be an oriented link with n components. Then

$$H[P](L) = (z/w)^{n-1}\Sigma_{k=1}^{n}\eta^{k-1}\widehat{E}_k\Sigma_{\pi}P(\pi L)$$

where the second summation is over all partition π of the components of L into k (unordered) subsets and $R(\pi L)$ denotes the product of the Homflypt polynomials of the k sublinks of L defined by π. Furthermore, $\widehat{E}_k = (\widehat{E}^{-1} - 1)(\widehat{E}^{-1} - 2)\cdots(\widehat{E}^{-1} - k + 1)$, with $\widehat{E} = E\frac{z}{w}$, $\widehat{E}_1 = 1$, and $\eta = (v - v^{-1})/w$.

Conjecture 4.5

Let $\mathcal{A} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z, r^{\pm 1}, s^{\pm 1}, w]$. Define binary operations $\circ, /$ by

$$a \circ b = pa + qb + z \text{ and } a * b = ra + sb + w.$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with the formula

$$a_n = (1 - z/(1-p-q))((1-p)/q)^{n-1} + z/(1-p-q).$$

Then $(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z, r^{\pm 1}, s^{\pm 1}, w], \circ, /, *, //, \{a_n\}_{n=1}^{\infty})$ is a Conway algebra and

$$\widetilde{W}(L) = (r/p)^{n-1} \Sigma_{k=1}^{n} \eta^{k-1} \widehat{E}_k \Sigma_{\pi} P(\pi L) + f(z, w),$$

for some polynomial $f(z, w)$.

Question

- For any generalized Conway algebra,

$$\widetilde{W}(L) = \Sigma_{k=1}^{n} A_k \Sigma_\pi W(\pi L),$$

for coefficients A_k, which depend on k?

Answer I guess NOT, because we cannot be sure that

$$\widetilde{W}(L_1 \sqcup L_2) = \widetilde{W}(L_1)\widetilde{W}(L_2).$$

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Thank you