NEW INVARIANTS OF KNOTOIDS

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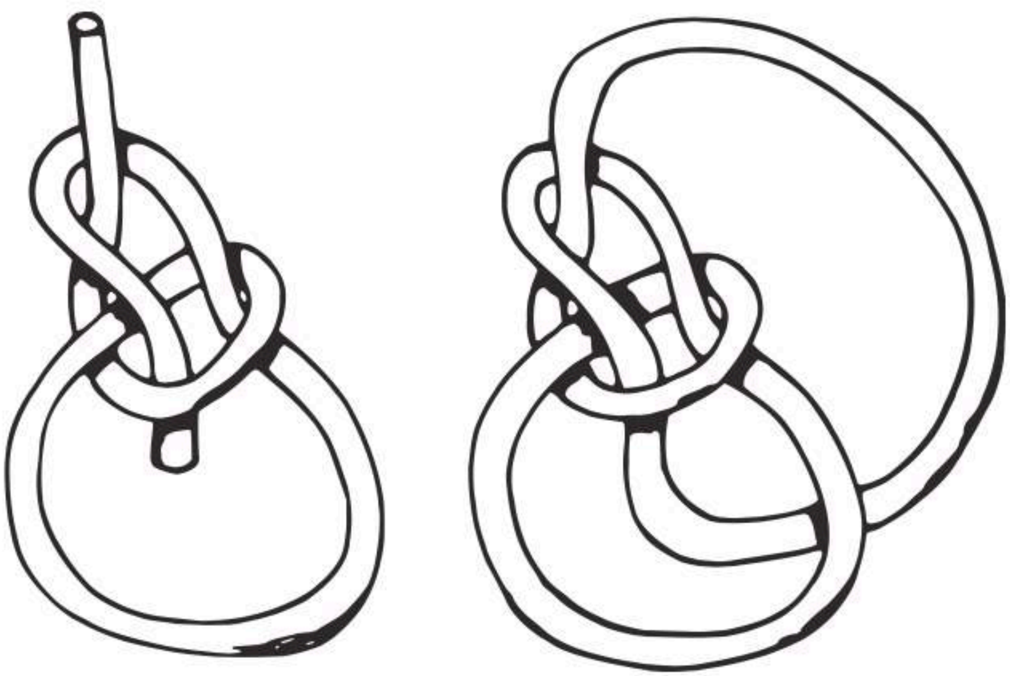
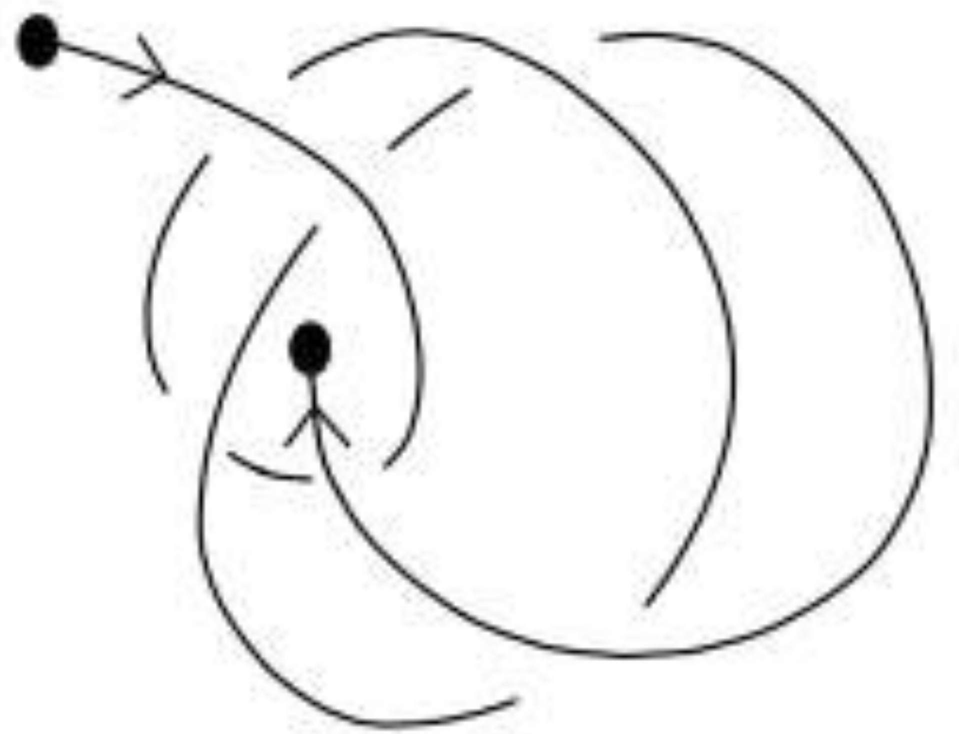


A knotoid (V. Turaev) is a knot diagram with two ends.
The ends can be in different regions of the diagram.
We study knotoids up to Reidemeister moves.
The moves do not move arcs across the ends of
the diagrams.



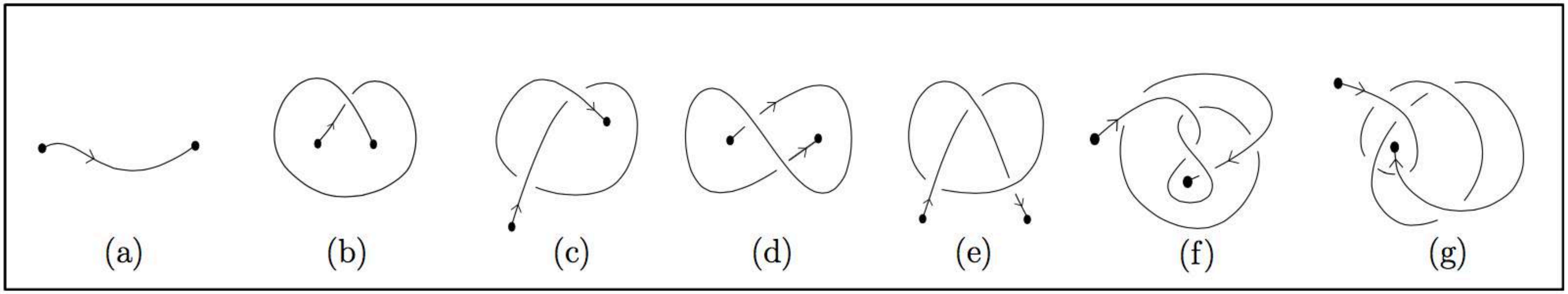
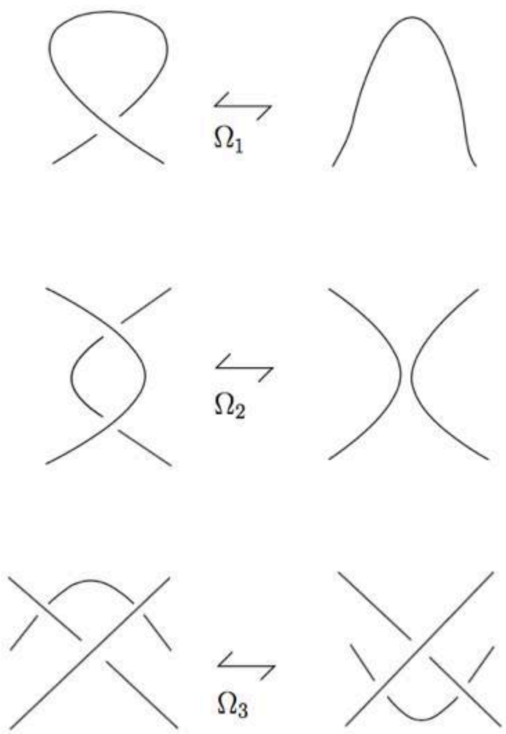
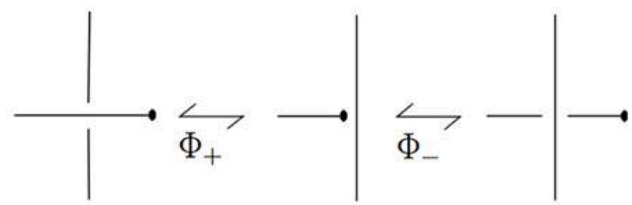


FIGURE 1. **Knotoid diagrams**

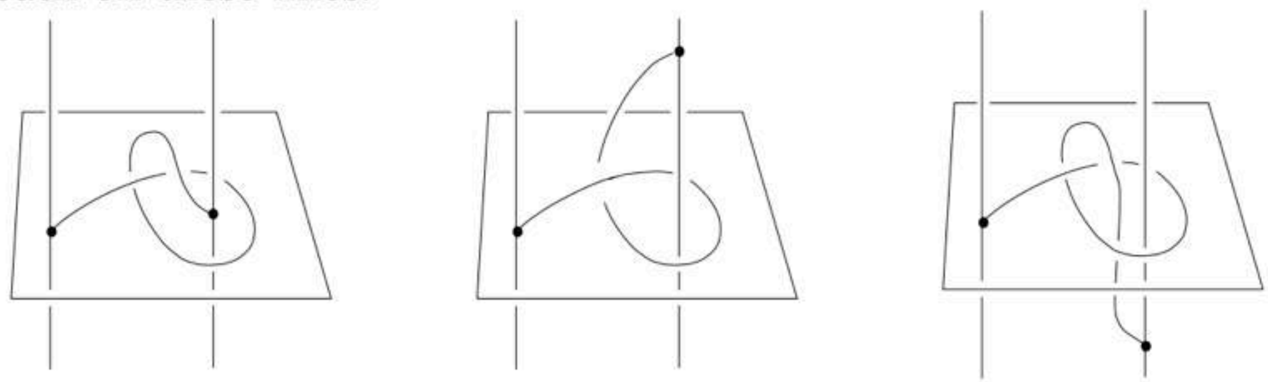


(a) $\Omega_{i=1,2,3}$- **moves**



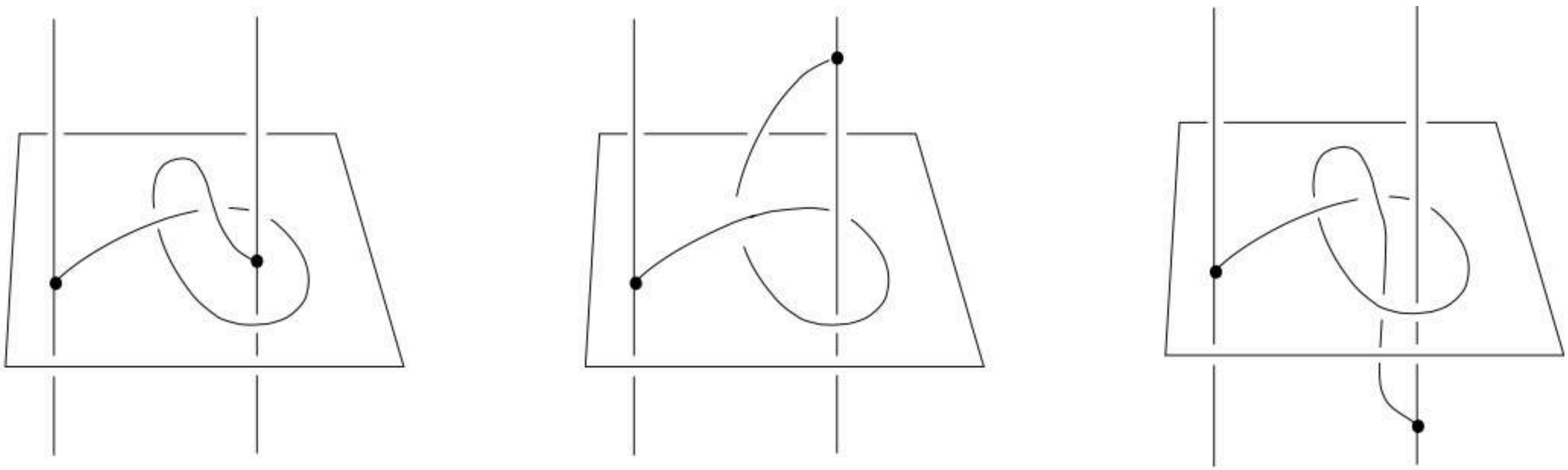
(b) **Forbidden knotoid moves**

2.1. **An Interpretation of Classical Knotoids in 3-Dimensional Space.** Let K be a knotoid diagram in $\mathbb{R}^2$. The plane of the diagram is identified with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. K can be embedded into $\mathbb{R}^3$ by pushing the overpasses of the diagram into the upper half-space and the underpasses into the lower half-space in the vertical direction. The tail and the head of the diagram are attached to the two lines, $\mathrm{t}\times\mathbb{R}$ and $\mathrm{h}\times\mathbb{R}$ that pass through the tail and the head, respectively and is perpendicular to the plane of the diagram. Moving the endpoints of K along these special lines gives rise to embedded open oriented curves in $\mathbb{R}^3$ with two endpoints of each on these lines.



(a) (b) (c)

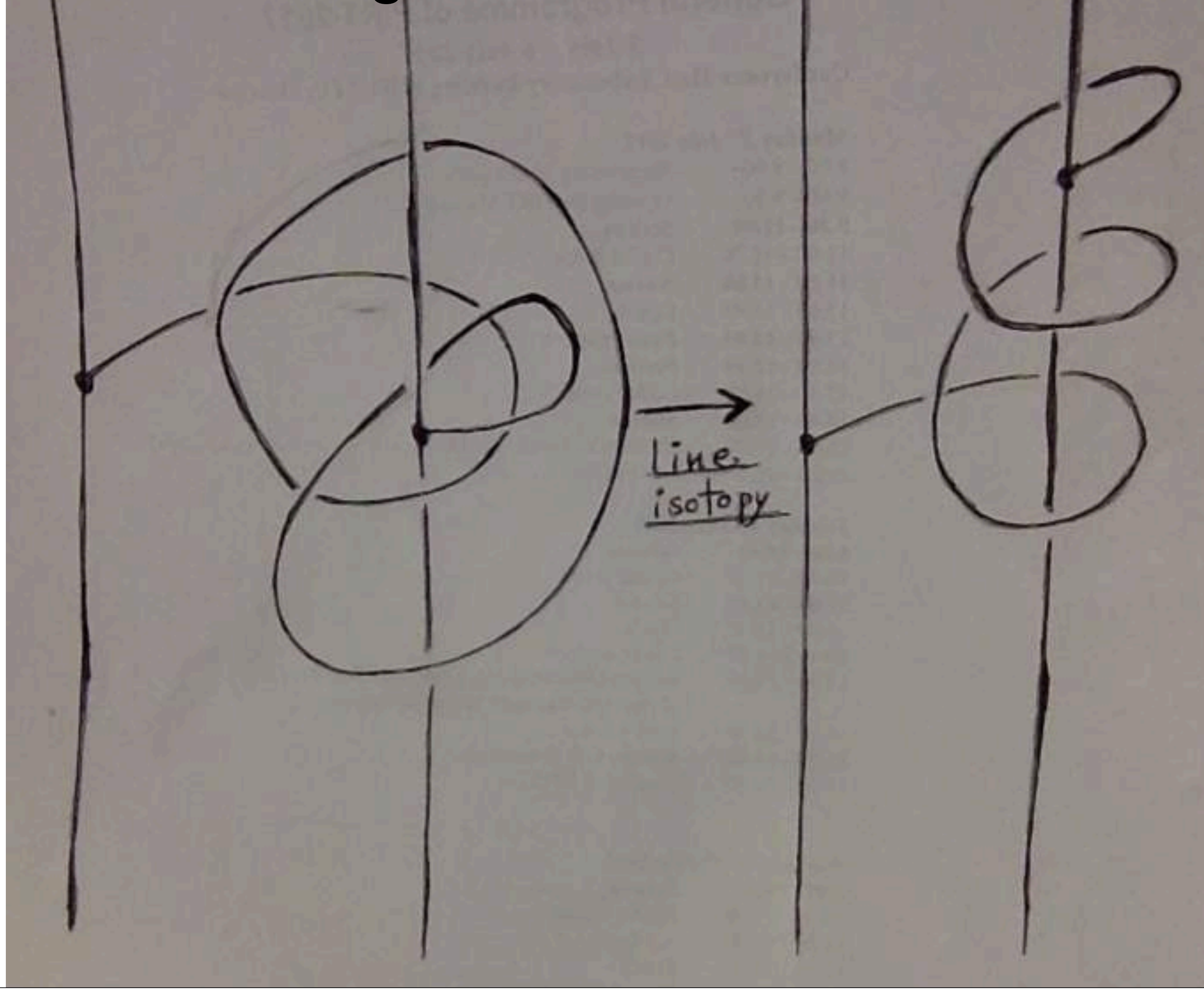
FIGURE 3. **Curves in $\mathbb{R}^3$ obtained by the knotoid diagram in Figure 1(c)**

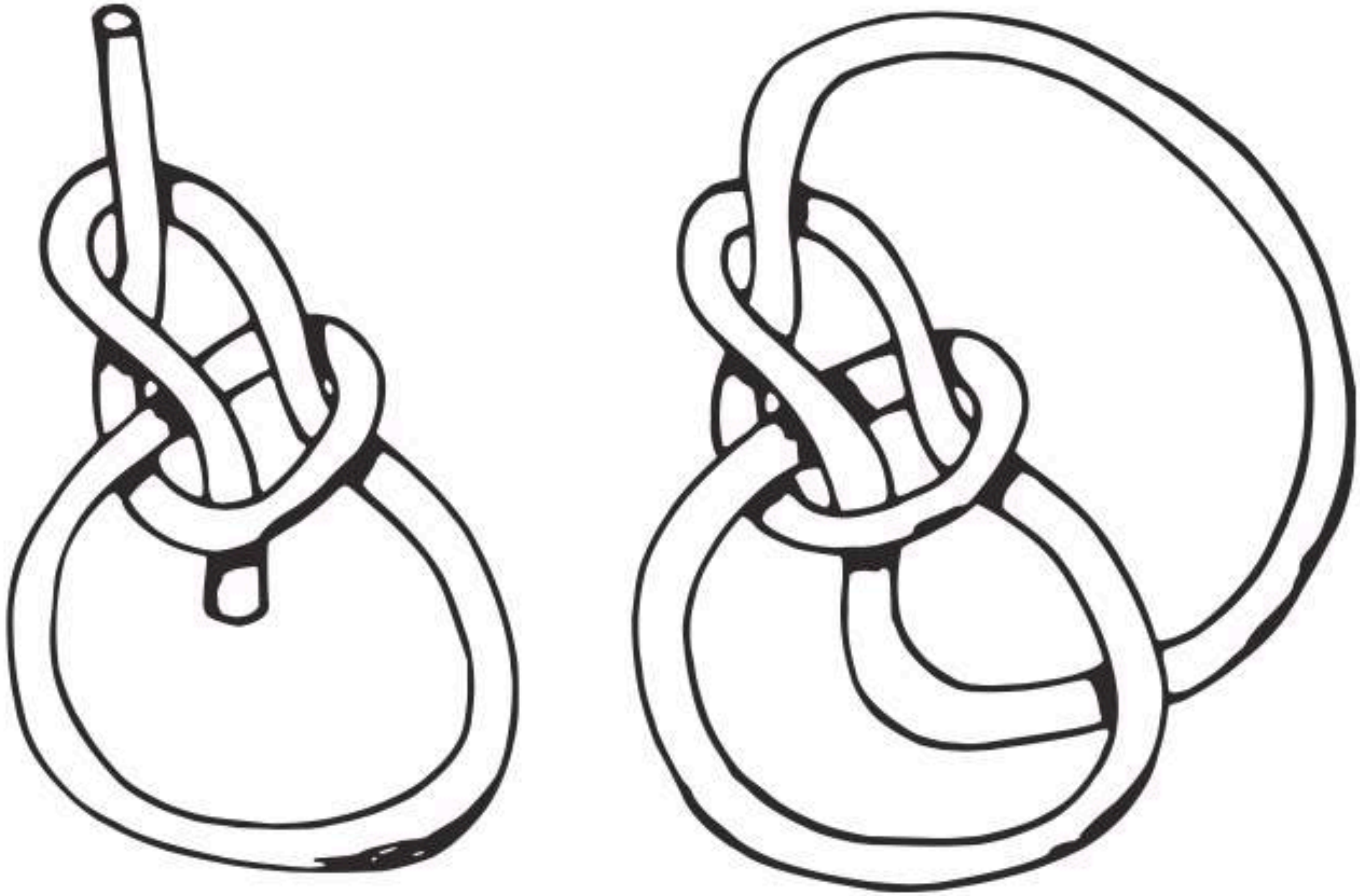


An embeded arc in R^3 becomes a knotoid on taking a generic projection to a plane.

Restricting isotopies of the arc to endpoint motions on the parallel lines (perpendicular to the plane) and otheswise in the complement of the two lines, preserves the knotoid type of the projection.

Descending Knotoid diagrams are unknotted.





Bowline as knotoid and an “underclosure” of the knotoid.

Studies of global and local entanglements of individual protein chains using planar and surface knotoids

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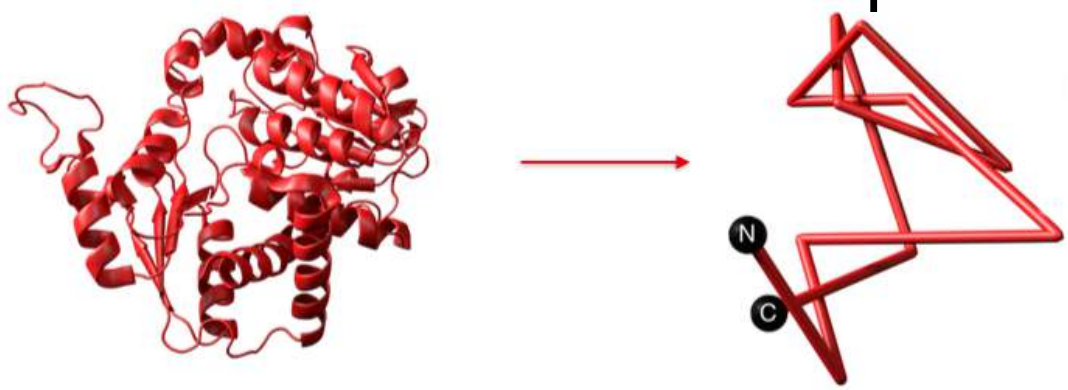
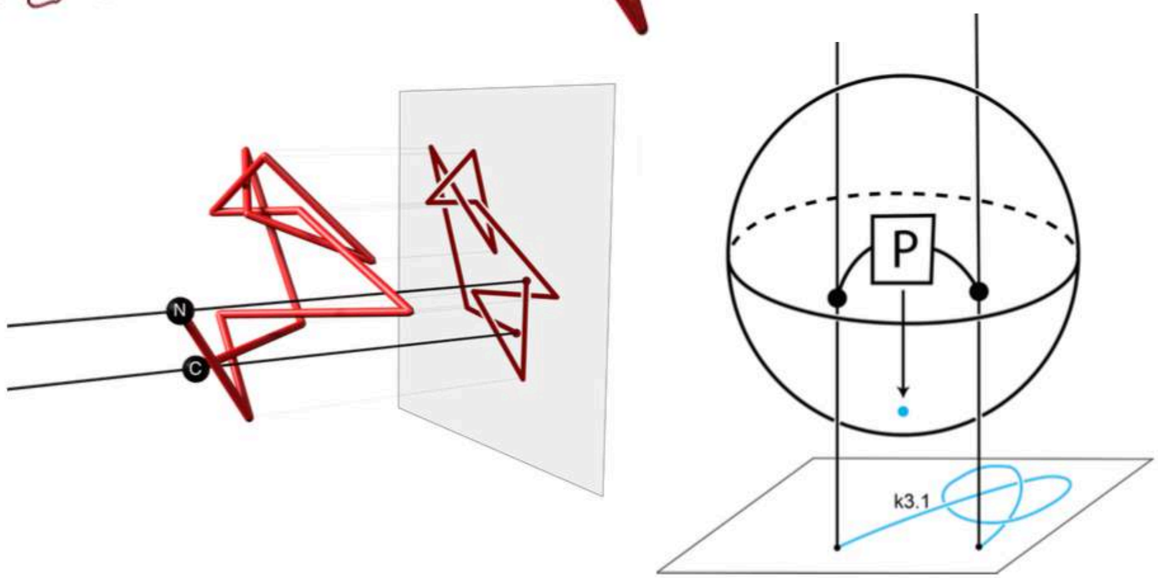
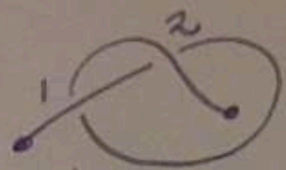
[4] To whom correspondence should be addressed. Email: andrzej.stasiak@unil.ch

We will be writing a paper on protein folding topology with these authors.







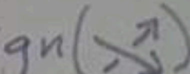
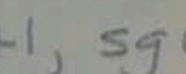
Flat Gauss Code

1212

Both crossings have odd parity

$i\,\alpha\beta\gamma\ldots\xi\,i$
$\alpha\beta\gamma\ldots\xi$: odd # of symbols

$J(K) =$ odd writhe
$= \sum_{c \in \text{Odd Crossings}} \mathrm{sgn}(c).$

sgn() = +1, sgn() = −1.

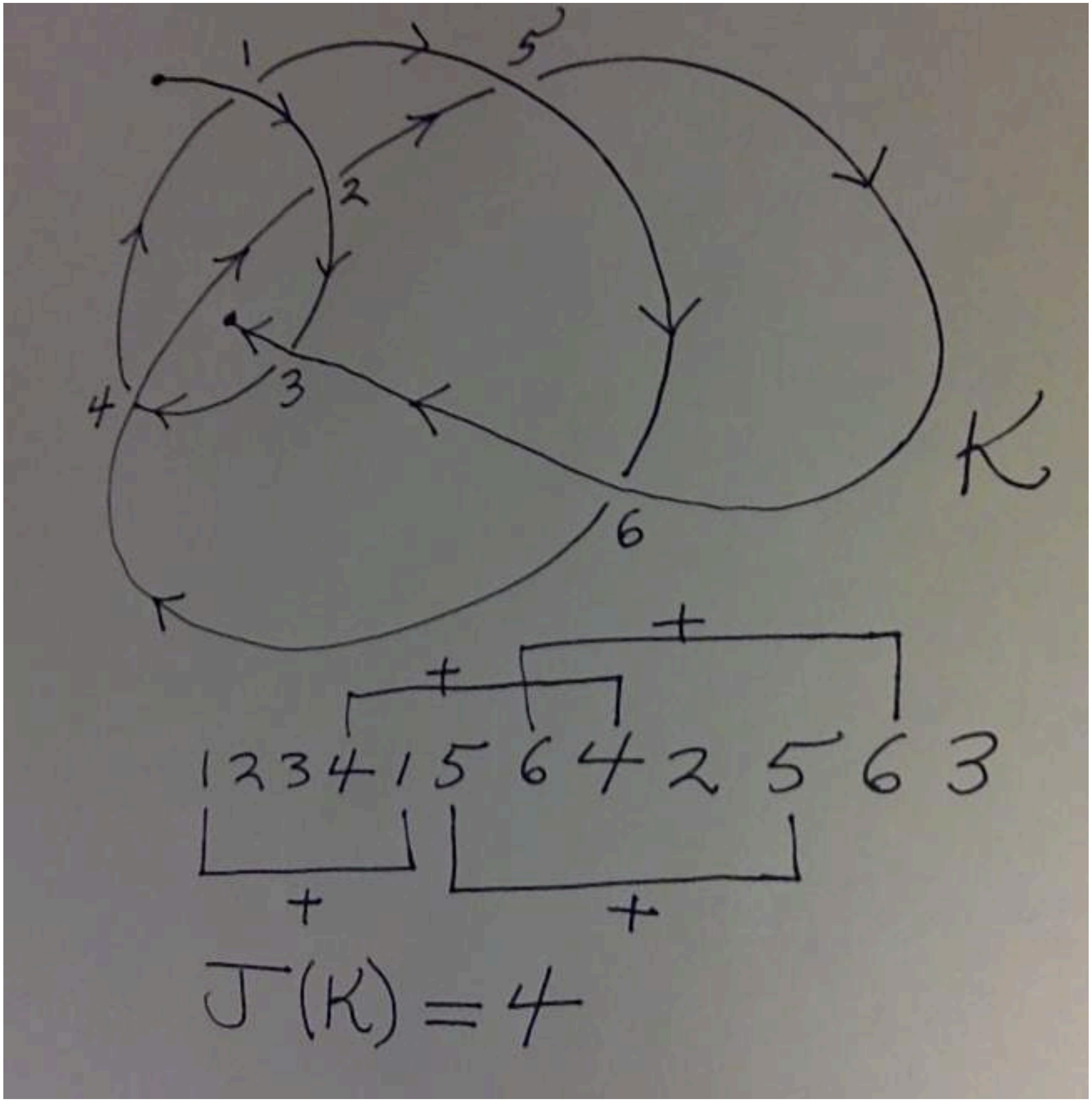
$J(K)$ is a knotoid invariant.

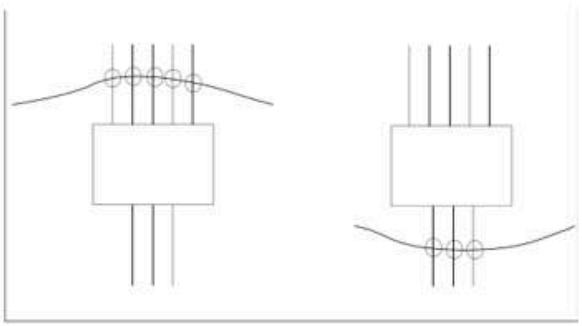
$J(K) = 0$ if K "standard" (endpts in same region).

$J(K^*) = -J(K)$

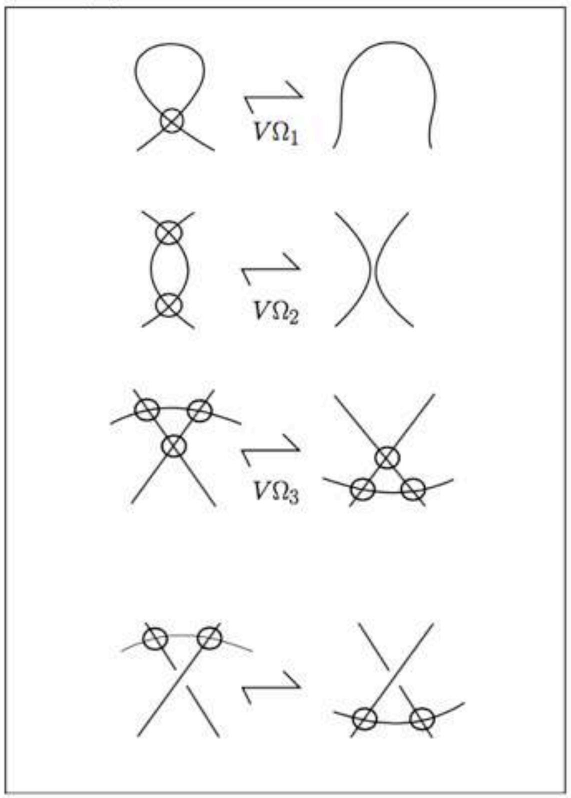
$J(K) \neq 0 \Rightarrow K$ non-standard.

Parity is a Fundamental Property of Knotoids

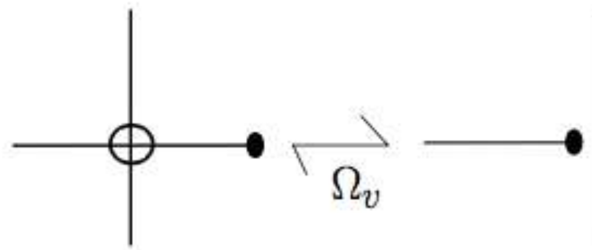

1
2
3
4
5
6
K
+
+
1 2 3 4 1 5 6 4 2 5 6 3
+
+
J(K) = 4



(a) **The detour move**



(b) **Virtual $\Omega_{i=1,2,3}$-moves and a partial virtual move**



Recall virtual knot theory.
We can have virtual knotoids
AND
we consider the virtual closure
of knotoids as method to study them.

Virtual Knot Theory
studies stabilized knots in thickened surfaces.

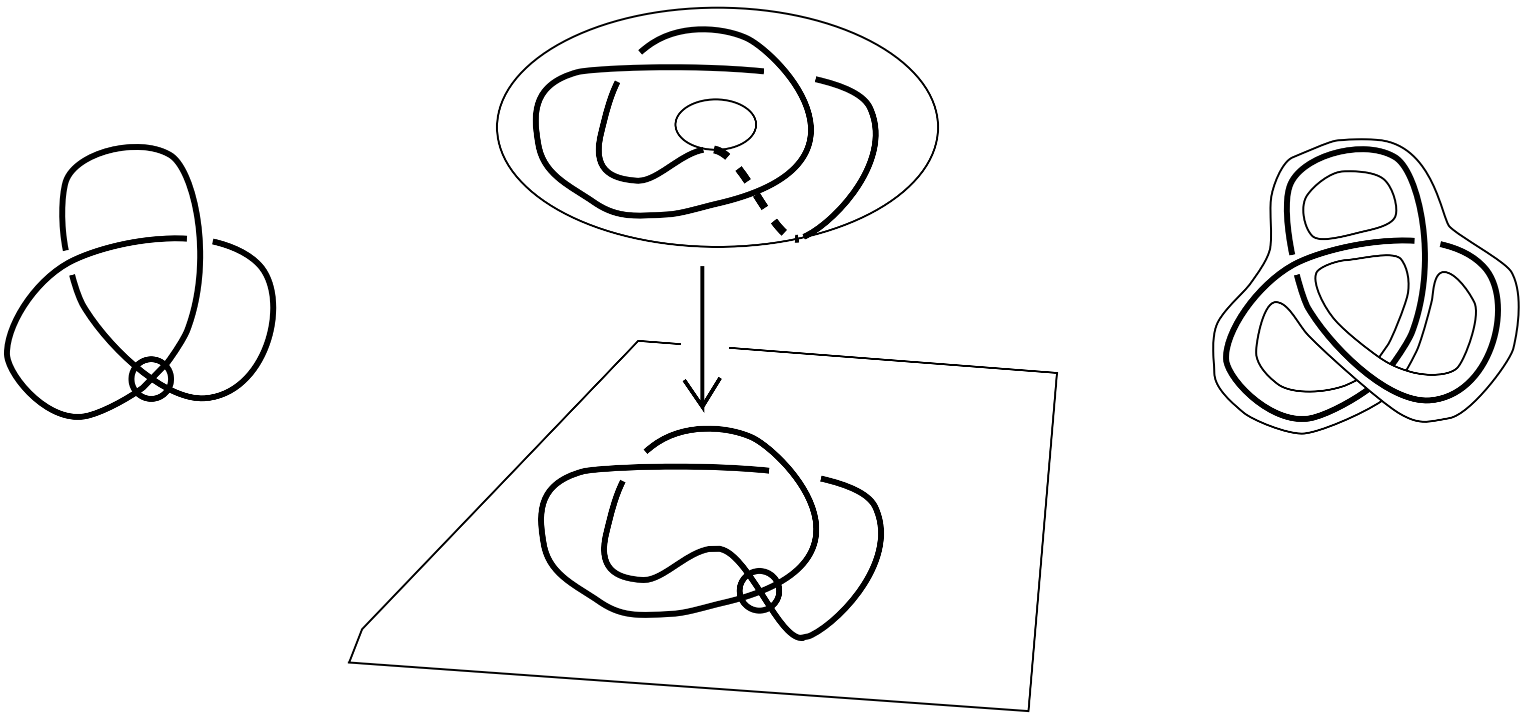
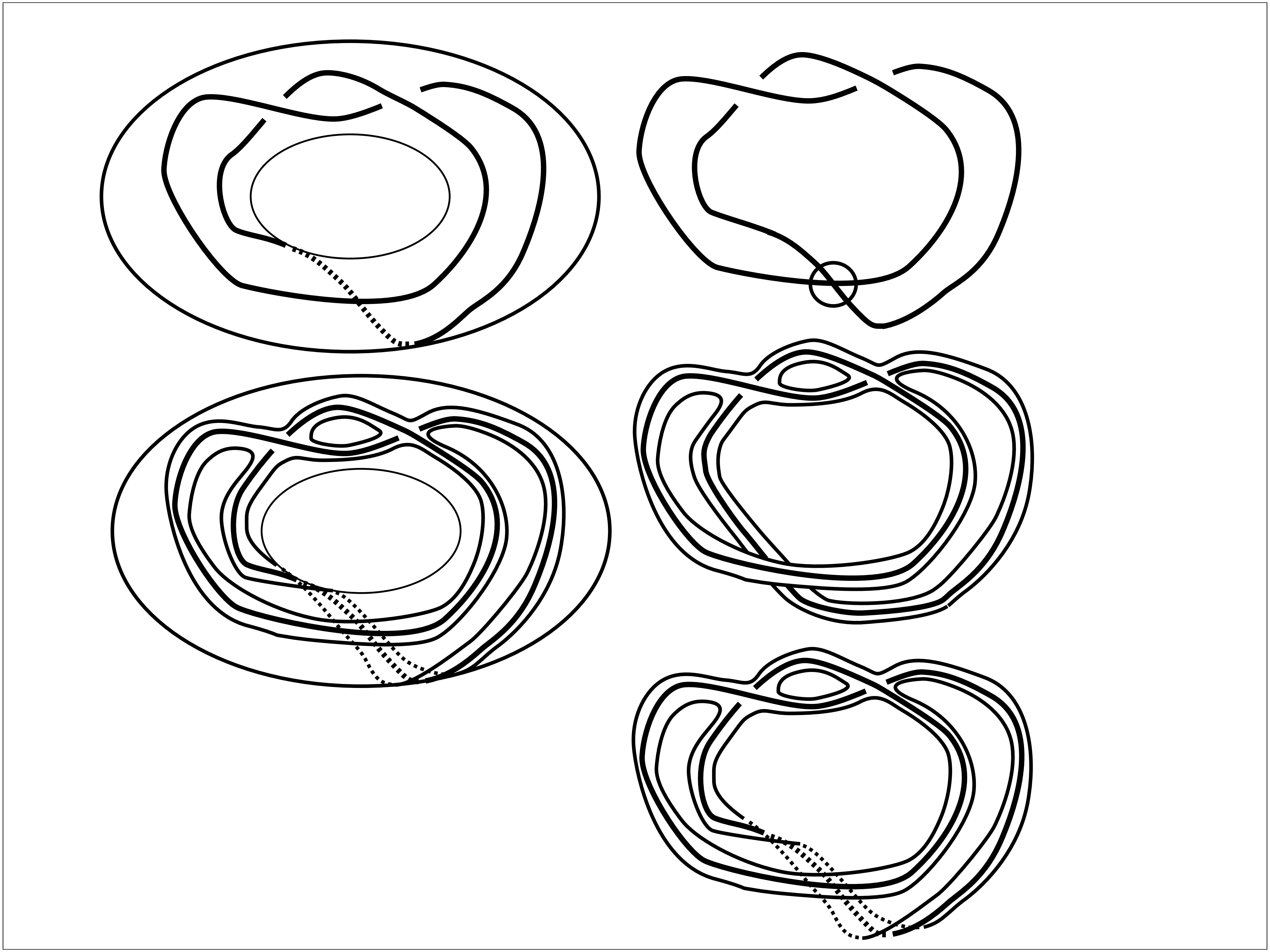
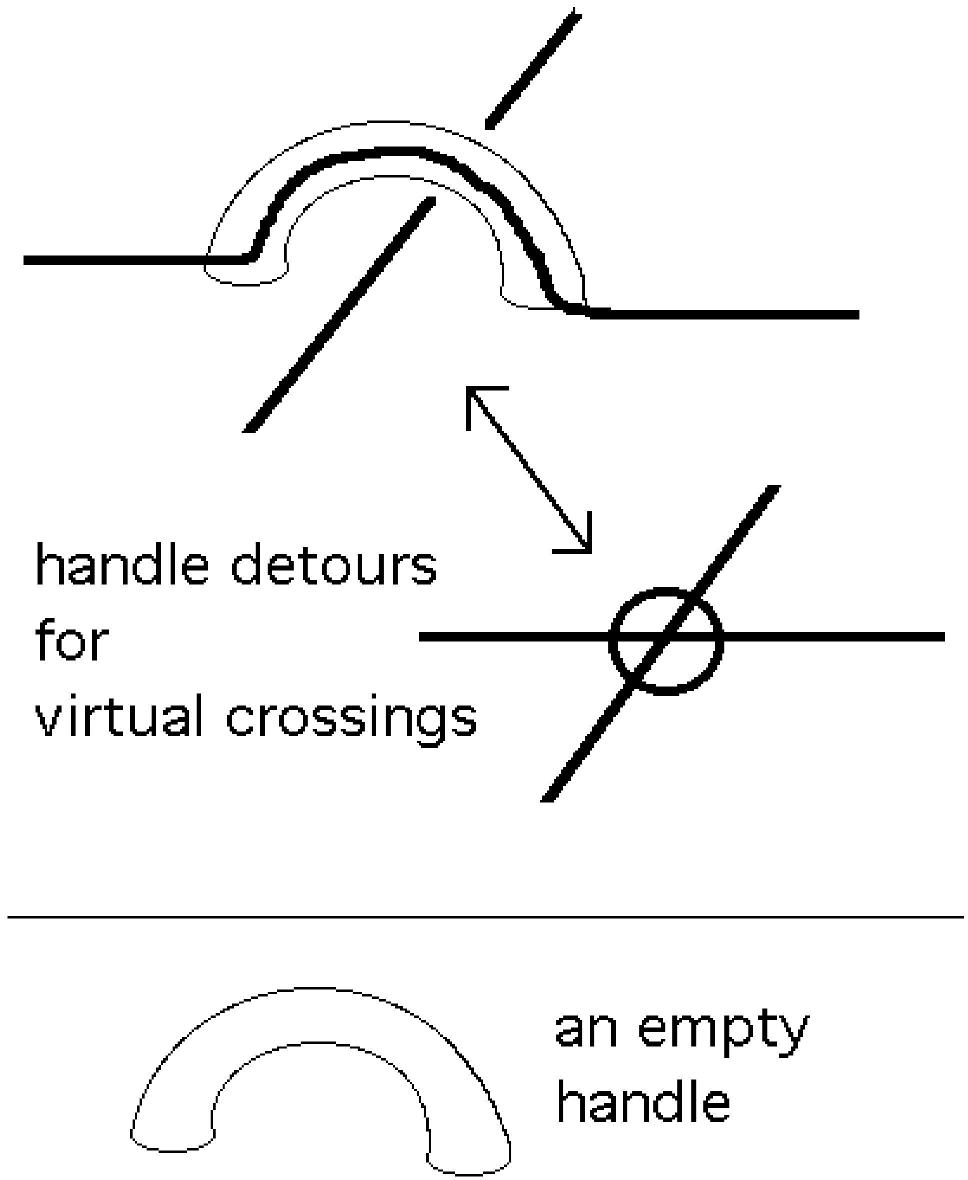
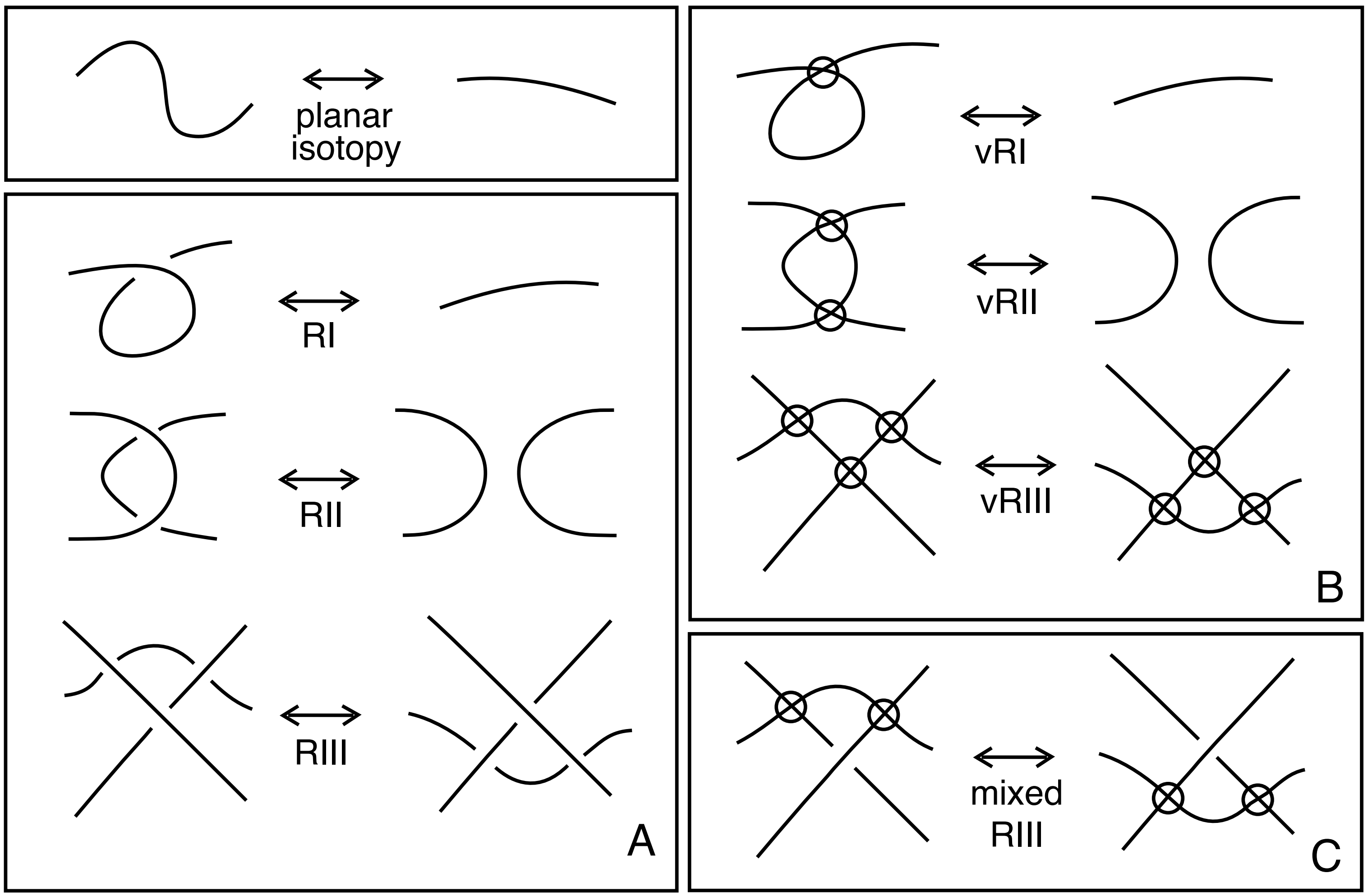


Figure 4: **Surfaces and Virtuals**

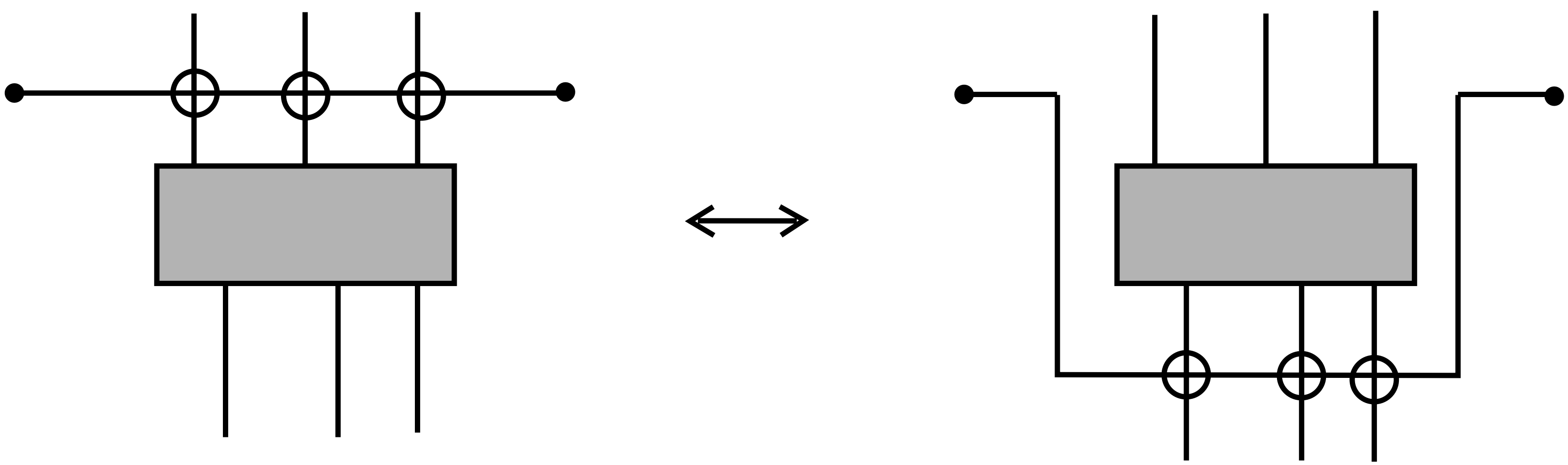



handle detours
for
virtual crossings
an empty
handle

Generalized Reidemeister Moves for Virtual Knots and Links



Detour Move



VKT

= Virtual Knot Theory

= Virtual Diagrams up to Virtual Equivalence

= Oriented Gauss Codes up to Reidemeister Moves

= Links in Thickened Surfaces up to 1-handle stabilization

$\overline{v}$: Knotoids in $S^2 \to$ Virtual knots of genus ≤ 1.

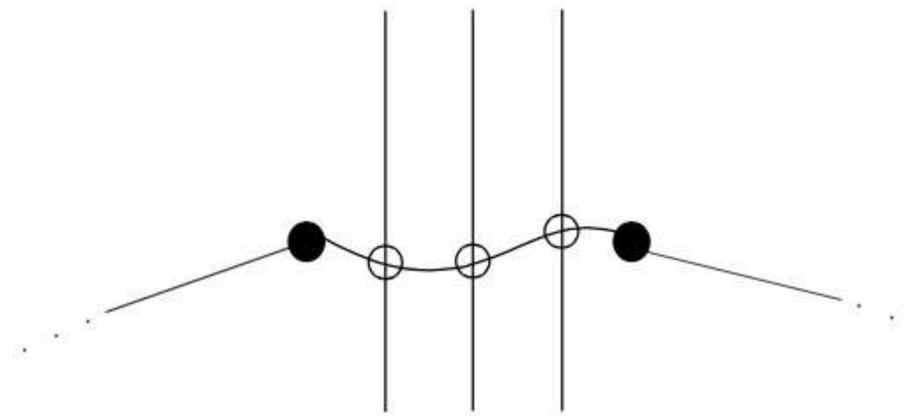


FIGURE 13. **The virtual closure of a knotoid diagram**

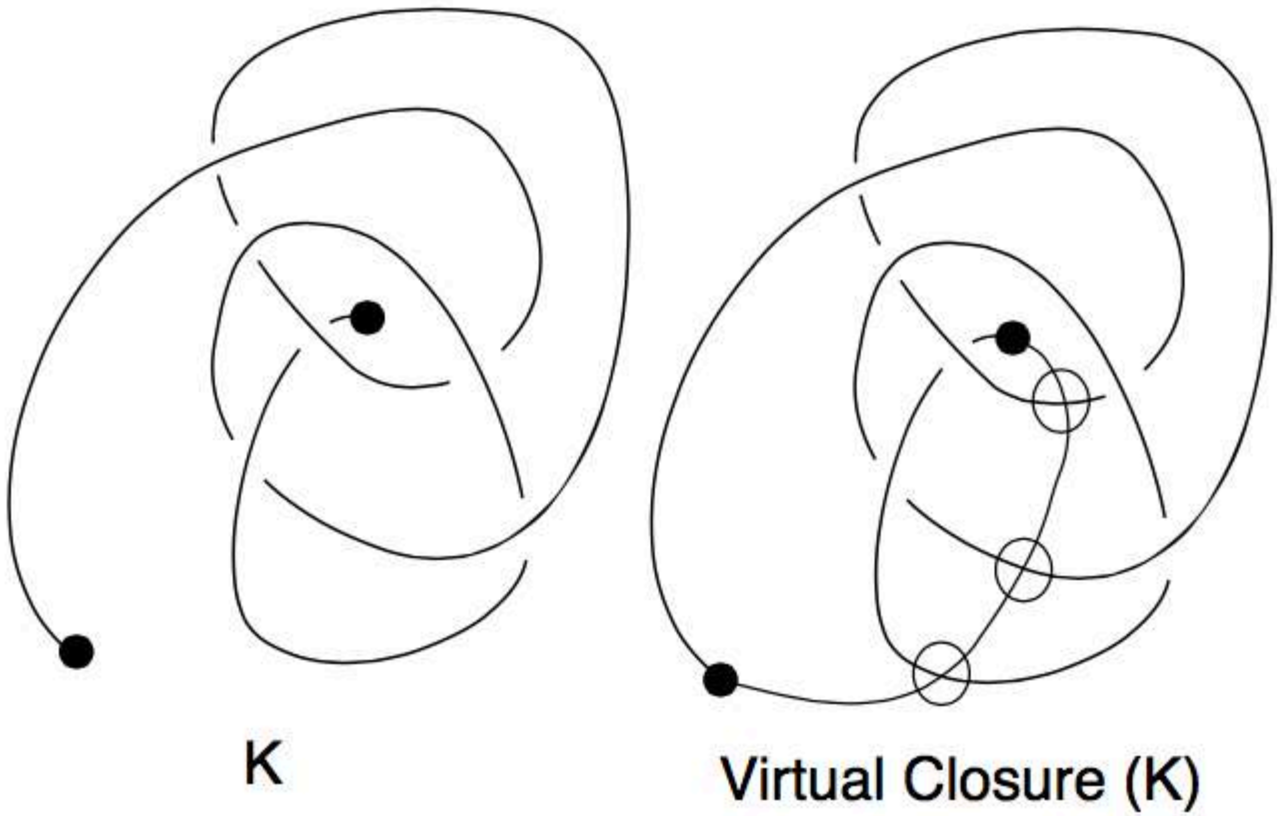
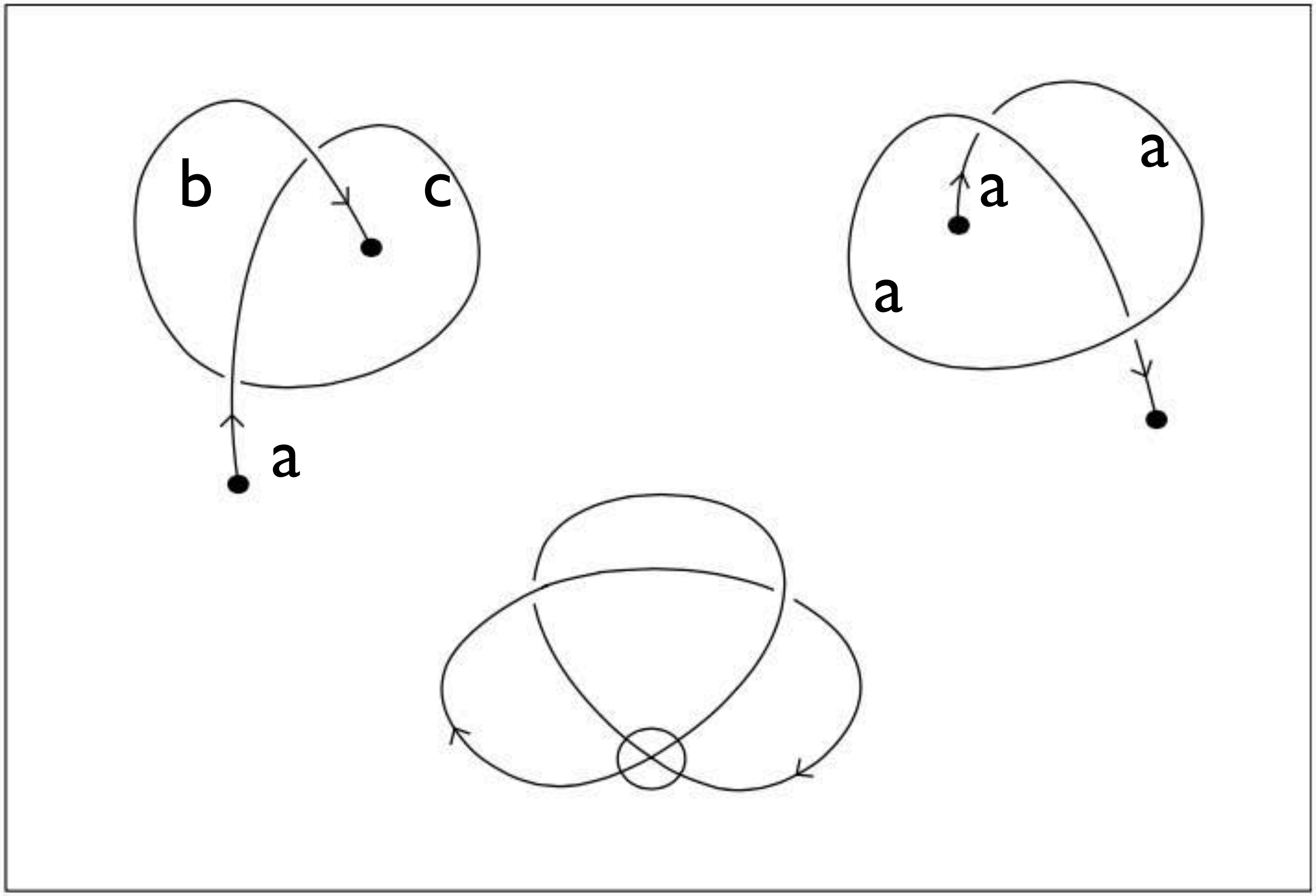


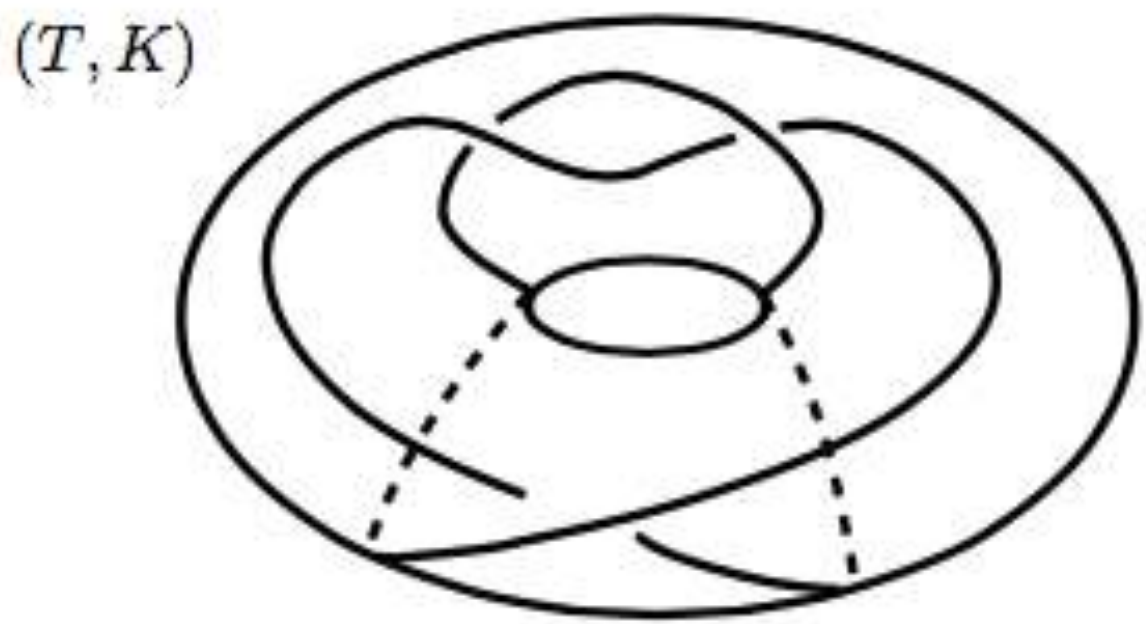
Figure 14: **Knotoid and Its Virtual Closure**

The virtual closure of a knotoid is supported in genus one (add a handle to the 2-sphere).

Two Distinct Knotoids with the same Virtual Closure



The virtual closure map is not surjective.



The virtual knot represented on the torus in the figure below is NOT in the image of the virtual closure map.

1.2. **The virtual closure map.** Let K be a knotoid diagram in S^2. The virtual knot diagram $\overline{v}(K)$ that is the virtual closure of K, lies in a torus when a handle is added to S^2 in a way that the connection arc α goes around the handle. The virtual knot diagram $\overline{v}(K)$ can be illustrated in an abstract way, as in Figure 1, where in the picture α denotes the connection arc going through the handle. Let $[a], [b]$ be the generators of $H_1(T^2)$. By the construction, the surface bracket states of the representation of $\overline{v}(K)$ in T^2 consist of isotopy classes of simple closed curves that are homologous to the curves $[a] + n[b]$ and $m[b]$, $n, m \in \mathbb{Z}$, for some choice of orientation assigned to state curves.

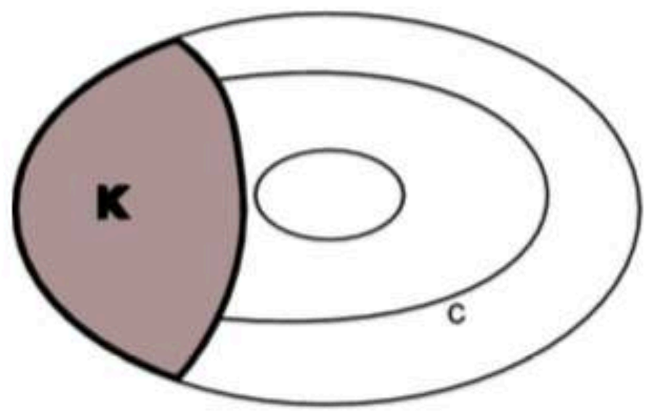


FIGURE 1. The virtual closure of a knotoid diagram in a torus

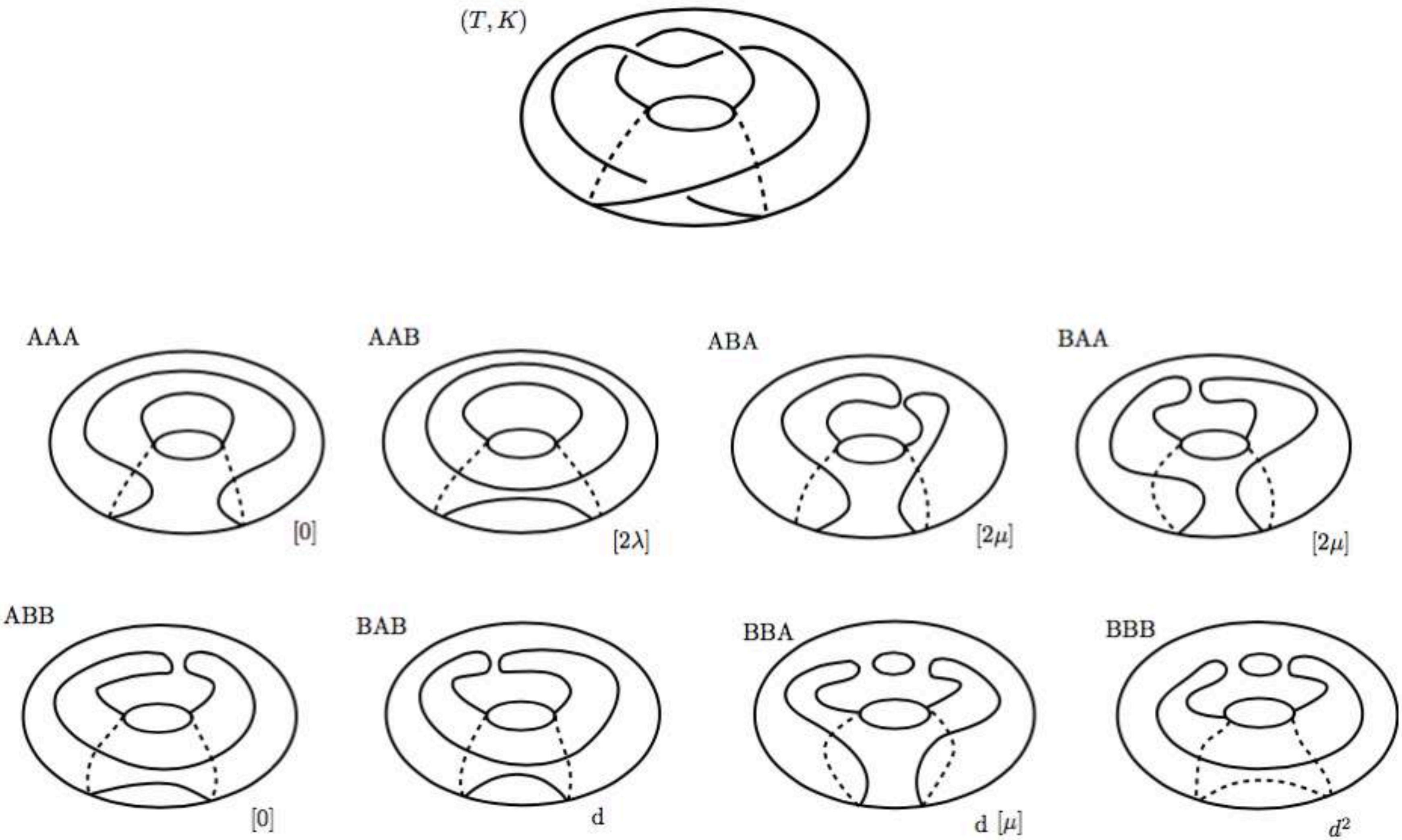
Lemma 1. *Let K be a virtual knot diagram in the class of a virtual knot k of genus 1 that is in the image of the virtual closure map $\overline{v}$ for knotoids. Let (T^2, k) be a representation of k in T^2. If the nontrivial isotopy classes of state curves of (T^2, k) are only of the form (for some choice of orientation) $k[a]$ and $m[b]$ for some $k, m \in \mathbb{Z} - \{0\}$, then $|k| = |m| = 1$.*

Proof. The isotopy classes of the state curves are taken up to orientation preserving self-homeomorphisms of T^2 that are not isotopic to identity map. As we mentioned above, at least one of the torus representations of k has a state curve of the form $[a]+n[b]$ when the state curves are oriented accordingly. And Kuperberg's theorem [] tells that the minimal representations of virtual knots are unique. These two facts imply that there exists an orientation preserving self-homeomorphism f of T^2 such that $f_*([a]+n[b]) = k[a]$ or $f_*([a]+n[b]) = m[b]$ where f_* is the induced isomorphism on $H_1(T^2, \mathbb{Z})$. The mapping class group of T^2 is isomorphic to the special linear group of 2×2 integral matrices, $SL(2, \mathbb{Z})$, then the map f can be represented by a matrix $M = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$ in $SL(2, \mathbb{Z})$. We have,

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix} \begin{pmatrix} 1 \\ n \end{pmatrix} = \begin{pmatrix} \mathrm{k} \\ 0 \end{pmatrix}$$

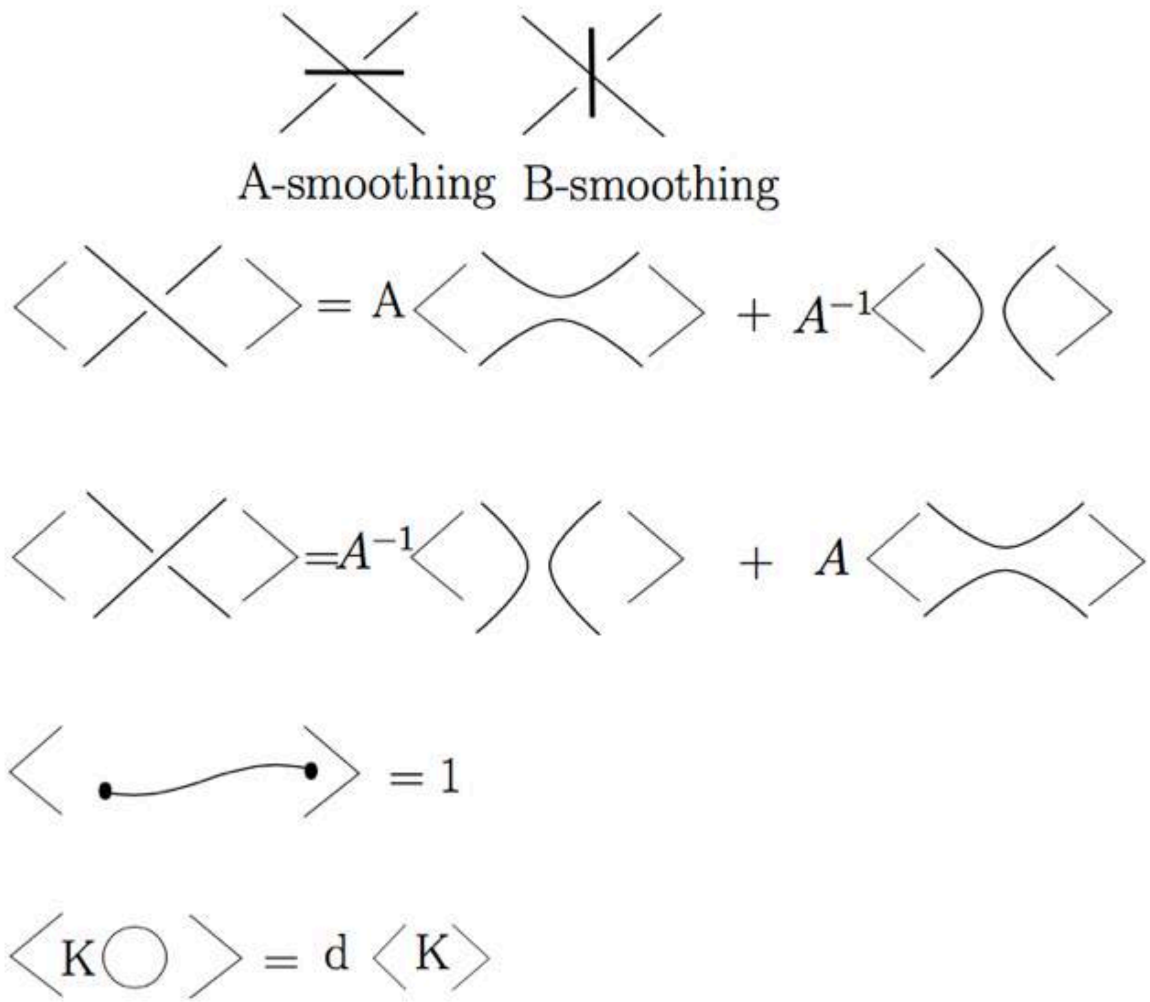
.

This is possible only if $R + nS = k$ and $T + nU = 0$. Equivalently, $n = -T/U$ and $R + (-T/U)S = 1/U = k$. We have $|U| = |k| = 1$ since $k \in \mathbb{Z} - \{0$. The second part of the proof for showing that $|m| = 1$, follows similarly. □

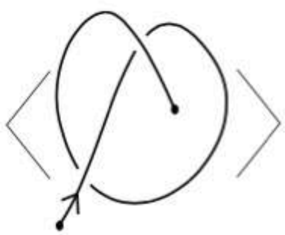
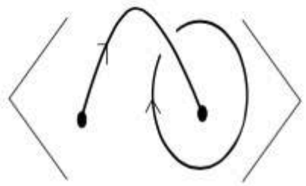


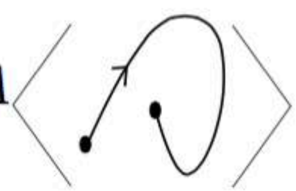
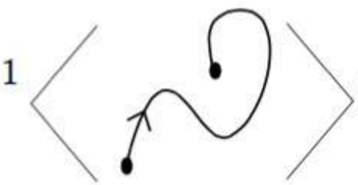
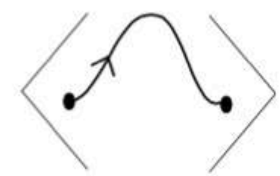
The states contradict the needs of the Lemma if the knot were in the image of the closure map.

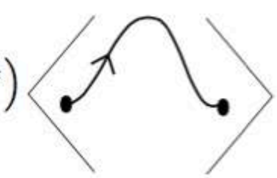
Bracket Polynomial for Knotoids



Bracket Calculation


$= A$

$+ A^{-1}$

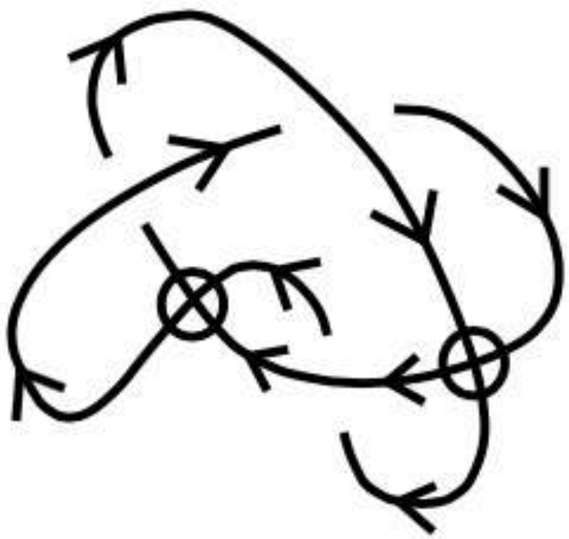

$= A(A$

$+ A^{-1}$

$) + A^{-4}$


$= (A^2 + 1 - A^{-4})$


Conjecture: The bracket polynomial detects the unknotted knotoid.

Discussion: This conjecture includes the well-known conjecture that the Jones polynomial detects the unknot.

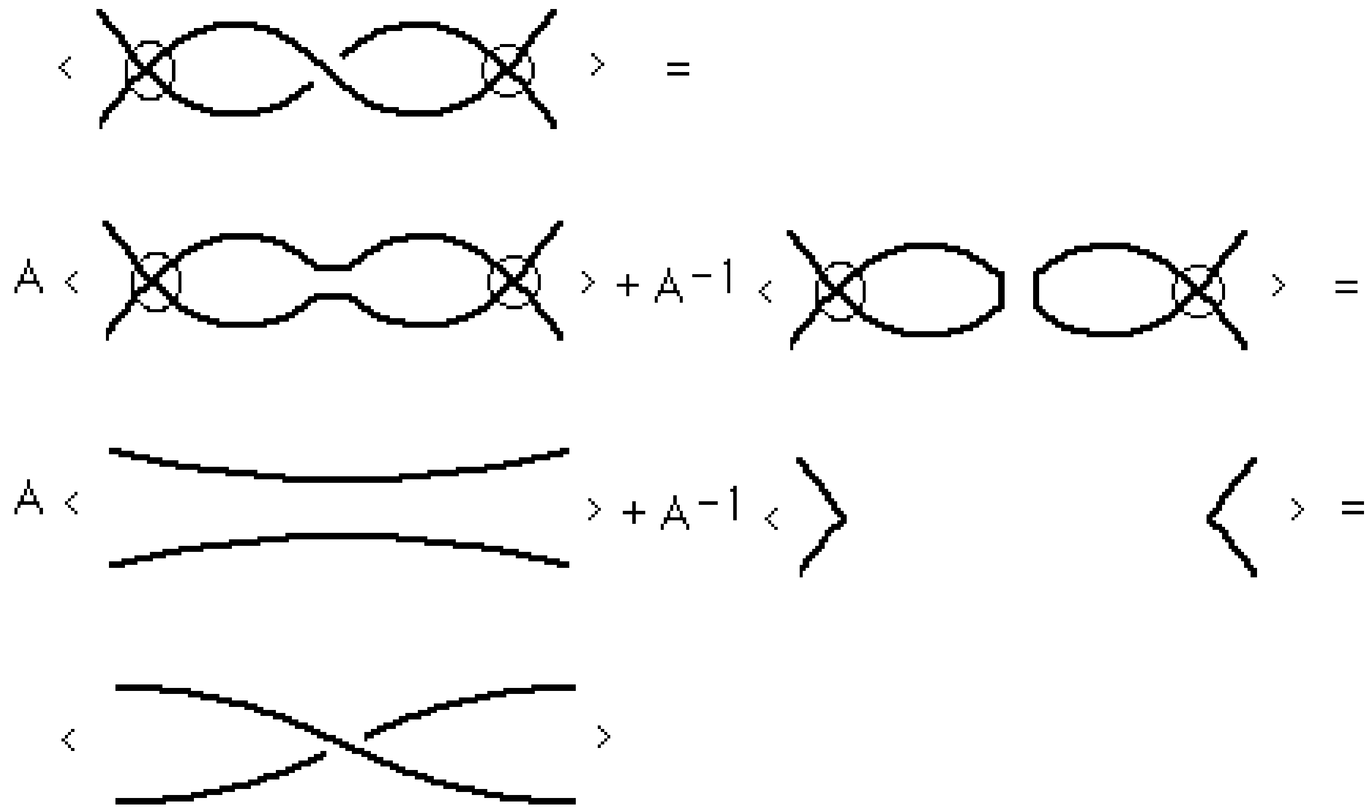
Note that the corresponding conjecture for virtual knots is false. There are non-trivial non-classical virtual knots with unit Jones polynomial. And there are examples of such virtual knots of genus one. This means that we conjecture that such virtual knots are not in the image of the closure map from knotoids.



Virtualized Trefoil Has Unit Jones Polynomial

We can prove that this virtual knot is not in the image of the closure map by examining isotopy classes of state loops on the torus.

Bracket Polynomial is Unchanged when smoothing flanking virtuals.



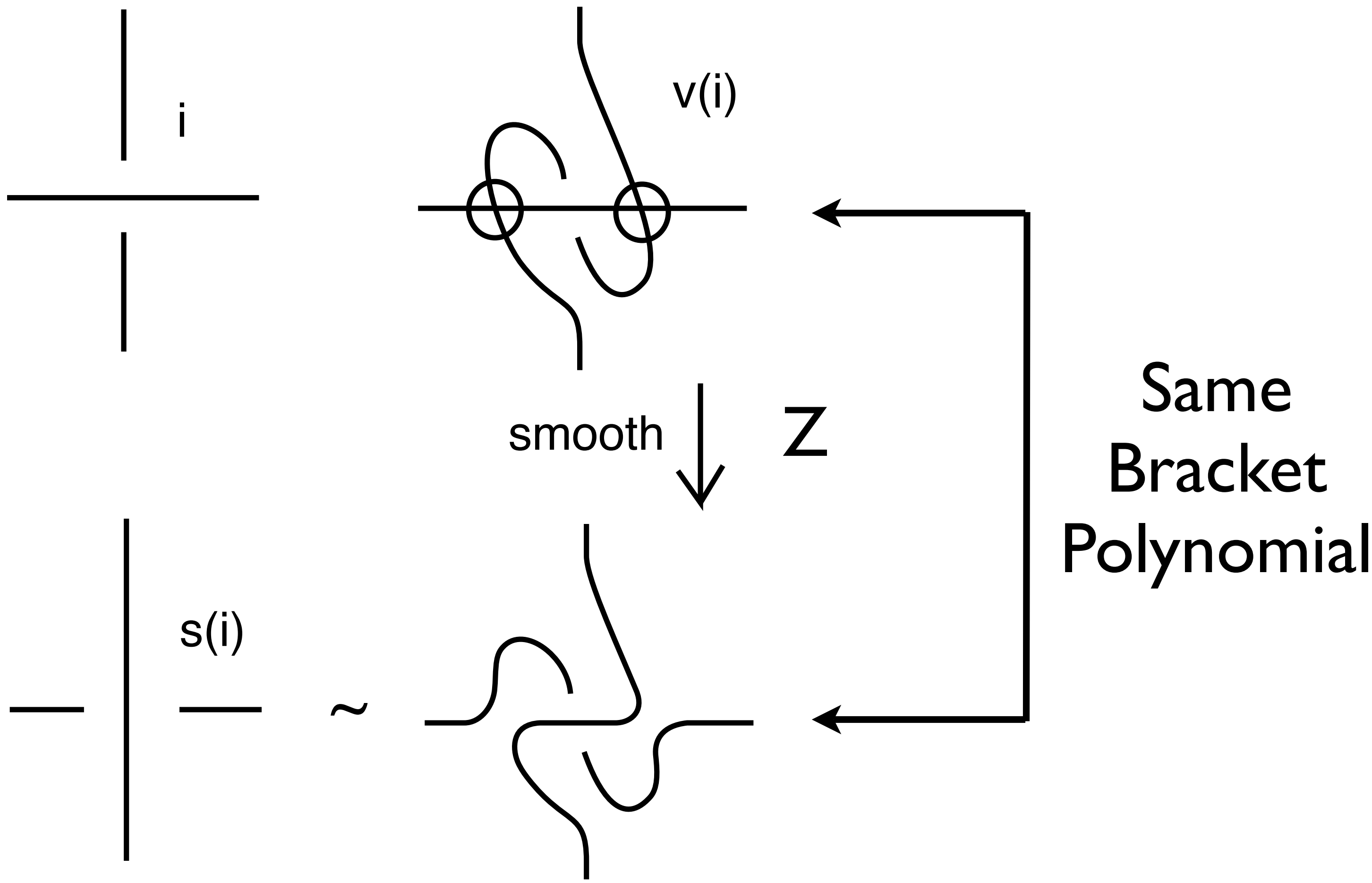


Figure 7. Switch and Virtualize

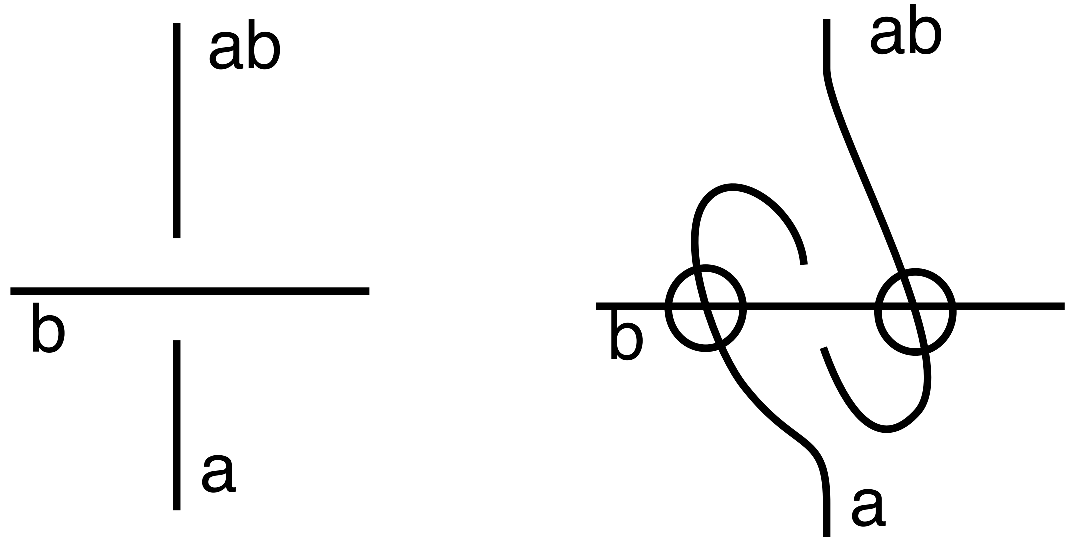
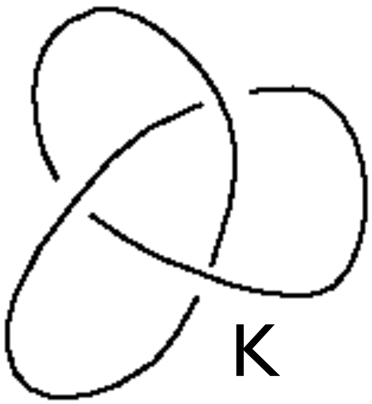


Figure 8. IQ(Virt)



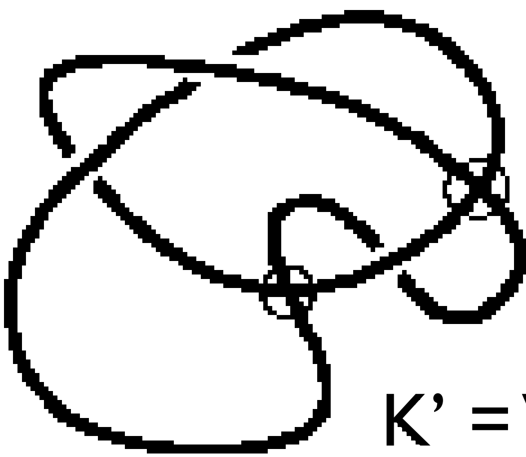
K

<Virt(K)> = <Switch(K)>

and

IQ(Virt(K)) = IQ(K).

There exist infinitely many non-trivial Virt(K) with unit Jones polynomial.



K' = Virt(K)

U

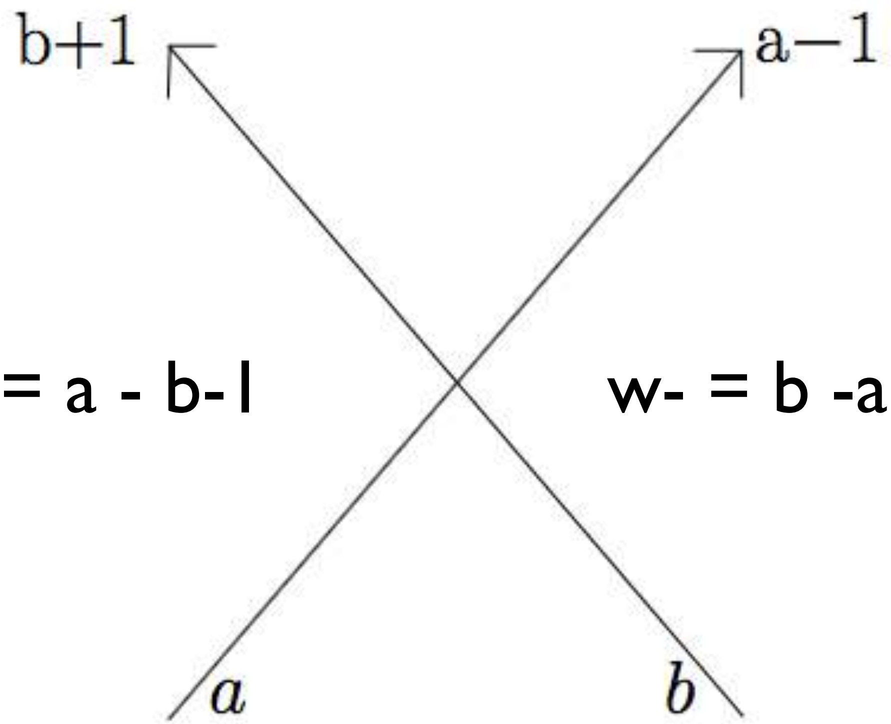
Nevertheless, we still hold to the

Conjecture: The bracket polynomial detects the unknotted knotoid.

Affine Index Polynomial for Knotoids

The Affine Index Polynomial is defined by knotoids both via virtual closure AND on its own grounds.

This invariant depends upon labeling a flat diagram with integers using the convention below.



$w+ = a - b - 1$

$w- = b - a + 1 = -w+$

Affine Index Polynomial

Let c be a classical crossing of K. We define two numbers at c resulting by the labeling of $F(K)$. These numbers that are denoted by $w_+(c)$ and $w_-(c)$, are defined as follows.

$$w_+(c) = a - (b+1)$$
$$w_-(c) = b - (a-1),$$

where a and b are the labels for the left and the right incoming arcs at the corresponding flat crossing to c, respectively. the numbers $w_+(c)$ and $w_-(c)$ are called *positive* and *negative* weights of c, respectively.
The *weight* of c is defined as

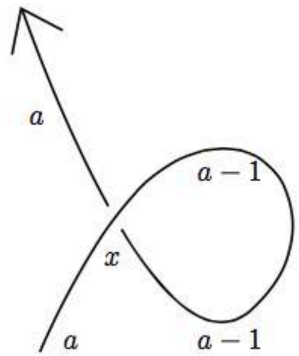
$$w_K(c) = \begin{cases} w_+(c), & \text{if the sign of } c \text{ is a positive crossing} \\ w_-(c), & \text{if the sign of } c \text{ is a negative crossing.} \end{cases}$$

Definition 12. The *affine index polynomial* of a virtual or classical knotoid diagram K is defined by the following equation.

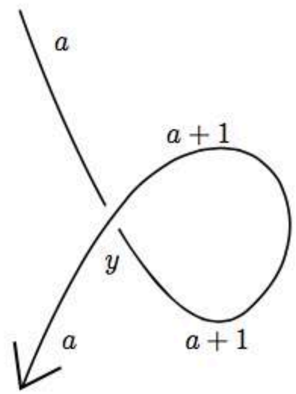
$$P_K(t) = \textstyle\sum_c \operatorname{sgn}(c)(t^{w_K(c)} - 1),$$

where the sum is taken over all classical crossings of a diagram of K and $\operatorname{sgn}(c)$ is the sign of c.

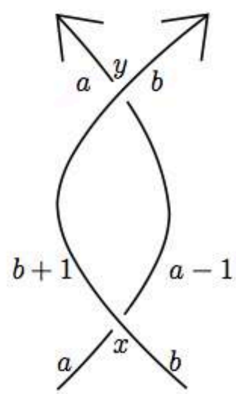
(related versions due to Henrich, Cheng, Dye for virtual knots and links)



$$w_+(x) = w_-(x) = 0$$



$$w_+(y) = w_-(y) = 0$$



$$w(x) = w_-(x) = b - a + 1$$
$$w(y) = w_+(y) = b - a + 1$$

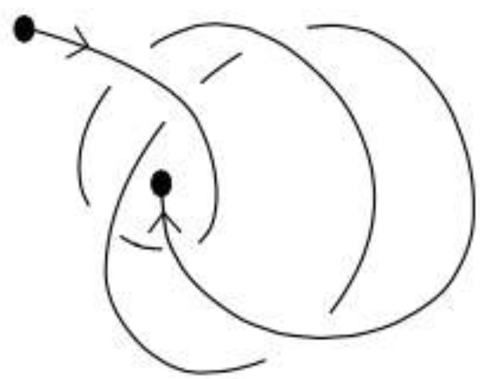
$$w_+(x) = c - b = w_+(x^*)$$
$$w_+(y) = a - c = w_+(y^*)$$
$$w_+(z) = b - a = w_+(z^*)$$

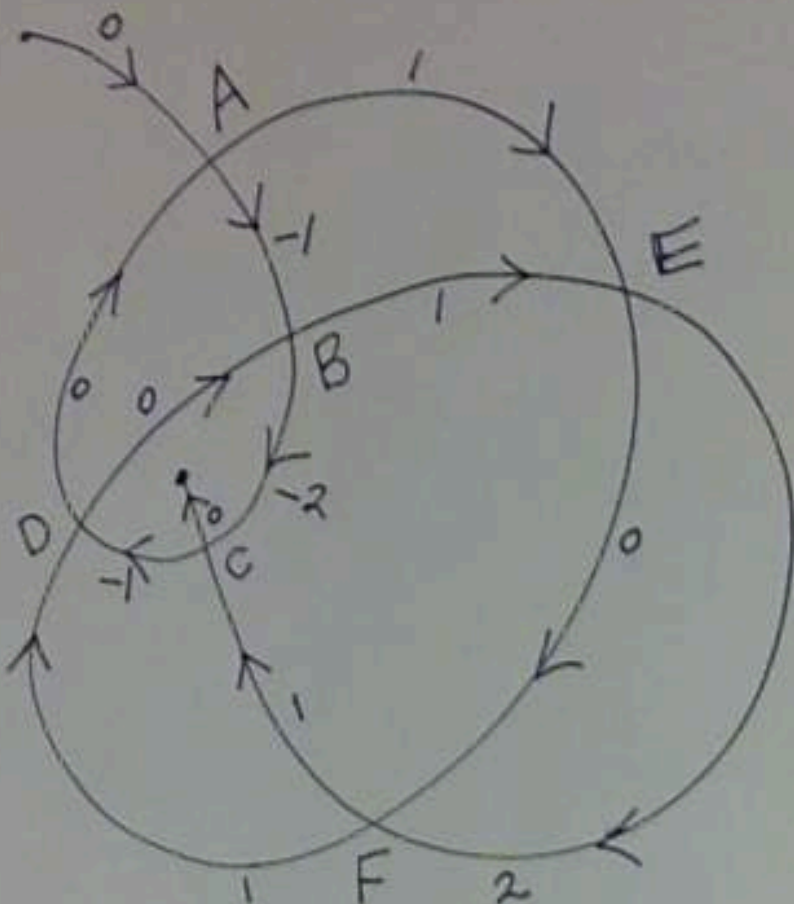
Theorem 4.10. *The affine index polynomial of a knotoid K in S^2 is symmetric with respect to $t \leftrightarrow t^{-1}$. Therefore, $P_K(t) = P_{\overline{K}}(t)$, where $\overline{K}$ denotes the inverse of K.*

4.5. **The height of a knotoid and the affine index polynomial.** The *height* (or the *complexity* with respect to Turaev's terminology in [37]) of a knotoid diagram in S^2 is the minimum number of crossings that a shortcut creates during the underpass closure. The *height of a knotoid* in S^2, K is defined as the minimum of the heights, taken over all equivalent classical knotoid diagrams to K and is denoted by $h(K)$. The height is an invariant of knotoids in S^2 [37]. A knotoid in S^2 is of knot-type if and only if its height is zero or equivalently a knotoid in S^2 has nonzero height if and only it is a proper knotoid [37].

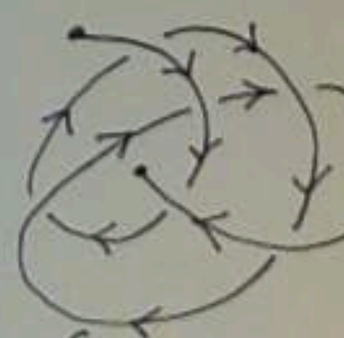
It is often hard to compute the height with an attempt of direct computation, for we should take into account all the equivalent knotoid diagrams. The affine index polynomial provides the following estimation for the height.

Theorem 4.12. *Let K be a knotoid in S^2.The height of K is greater than or equal to the maximum degree of the affine index polynomial of K.*





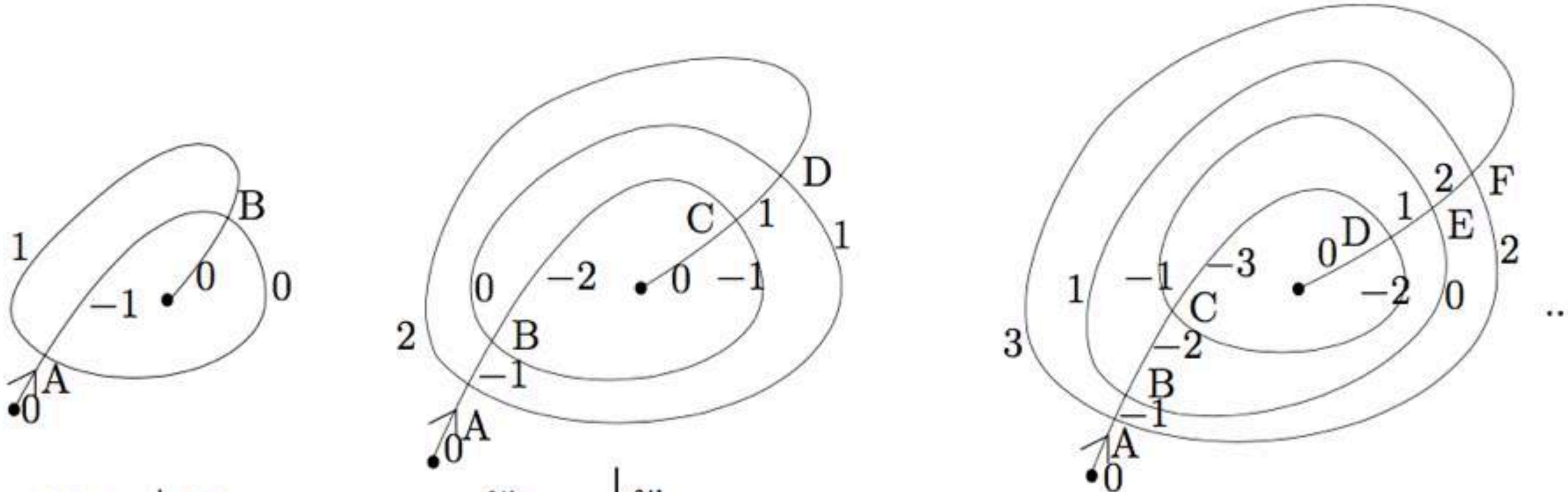
	w_+	w_-
A	-1	$+1$
B	-2	2
C	2	-2
D	1	-1
E	-1	1
F	1	-1



K

$$P_K = t^2 + t^{-2} + 2t + 2t^{-1} - 6$$

$$\Rightarrow ht(K) \geq 2 \Rightarrow ht(K) = 2.$$



	w_+	w_-
A	-1	$+1$
B	$+1$	-1

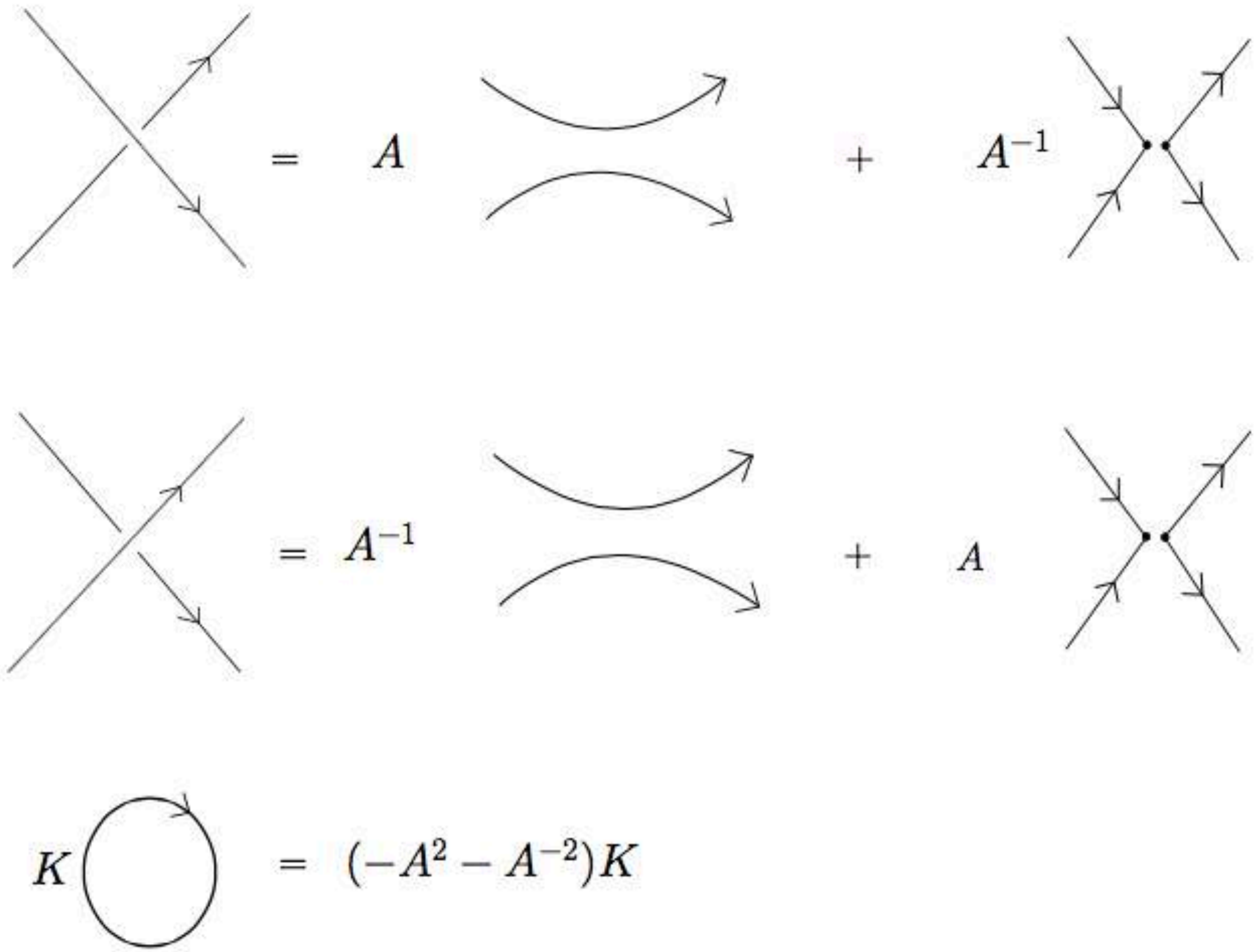
	w_+	w_-
A	-2	$+2$
B	-1	$+1$
C	$+2$	-2
D	$+1$	-1

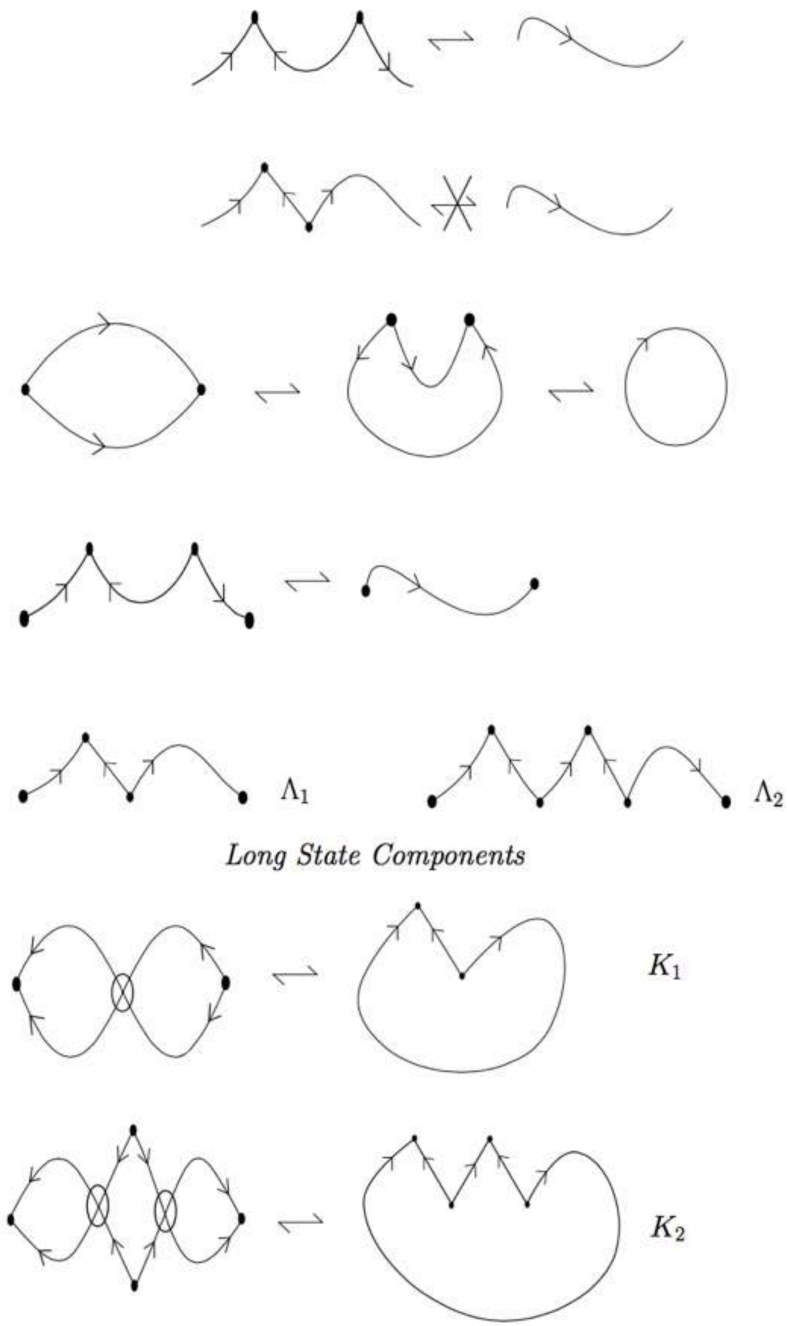
	w_+	w_-
A	-3	+3
B	-2	+2
C	-1	+1
D	+3	-3
E	+2	-2
F	+1	-1

FIGURE 35. **Flat spiral knotoid diagrams**

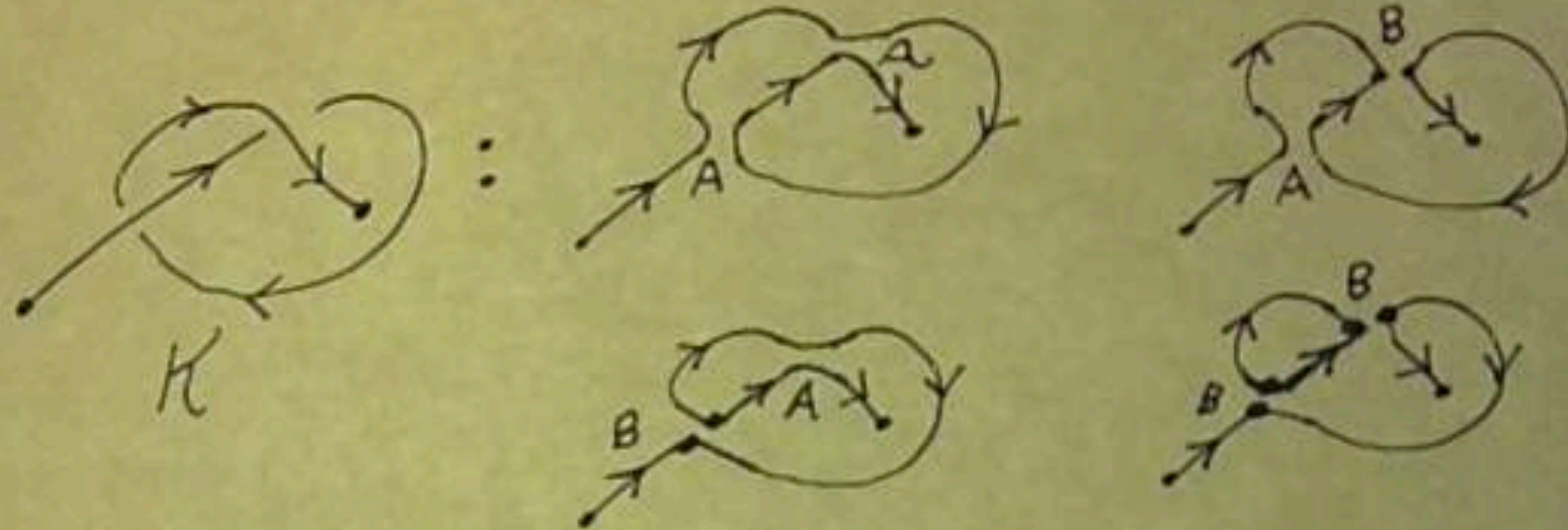
These give examples of heights n for any natural number n.

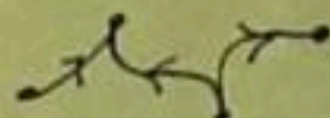
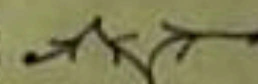
Arrow Polynomial for Knotoids





Cusp Rules for Arrow Polynomial



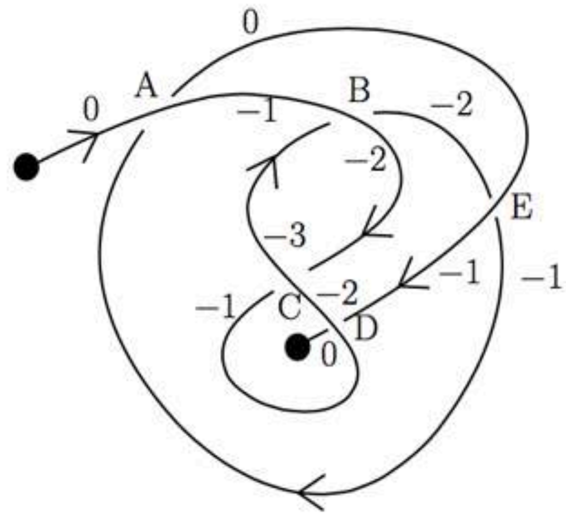
$$a(K) = A^2 + AB$$  $$+ AB$$  $$+ B^2\delta$$

$$= A^2 + 2\lambda_1 + A^{-2}(-A^2 - A^{-2})$$

$$= A^2 + 2\lambda_1$$

$$a(K) = (A^2 - 1 - A^{-4}) + 2\lambda_1$$

Example 5.4. The reader can easily see that the height of the knotoid diagram K given in Figure 1(f) is equal to 2. We want to find out if there exists an equivalent knotoid diagram to K with less height. The affine index polynomial of K is $P_K(t) = 2t + 2t^{-1} - 4$, as can be verified by Figure 41. The arrow polynomial of K is $A[K] = (-A^{-5} + 2A^{-1} - A^3 - A^7) + 2(A - A^5)\Lambda_1$. The affine index polynomial and the arrow polynomial both assure that the height of K is at least 1. Therefore we have, $1 \leq h(K) \leq 2$. This is a case where our tools discussed in this paper can not give an exact estimation for the height.

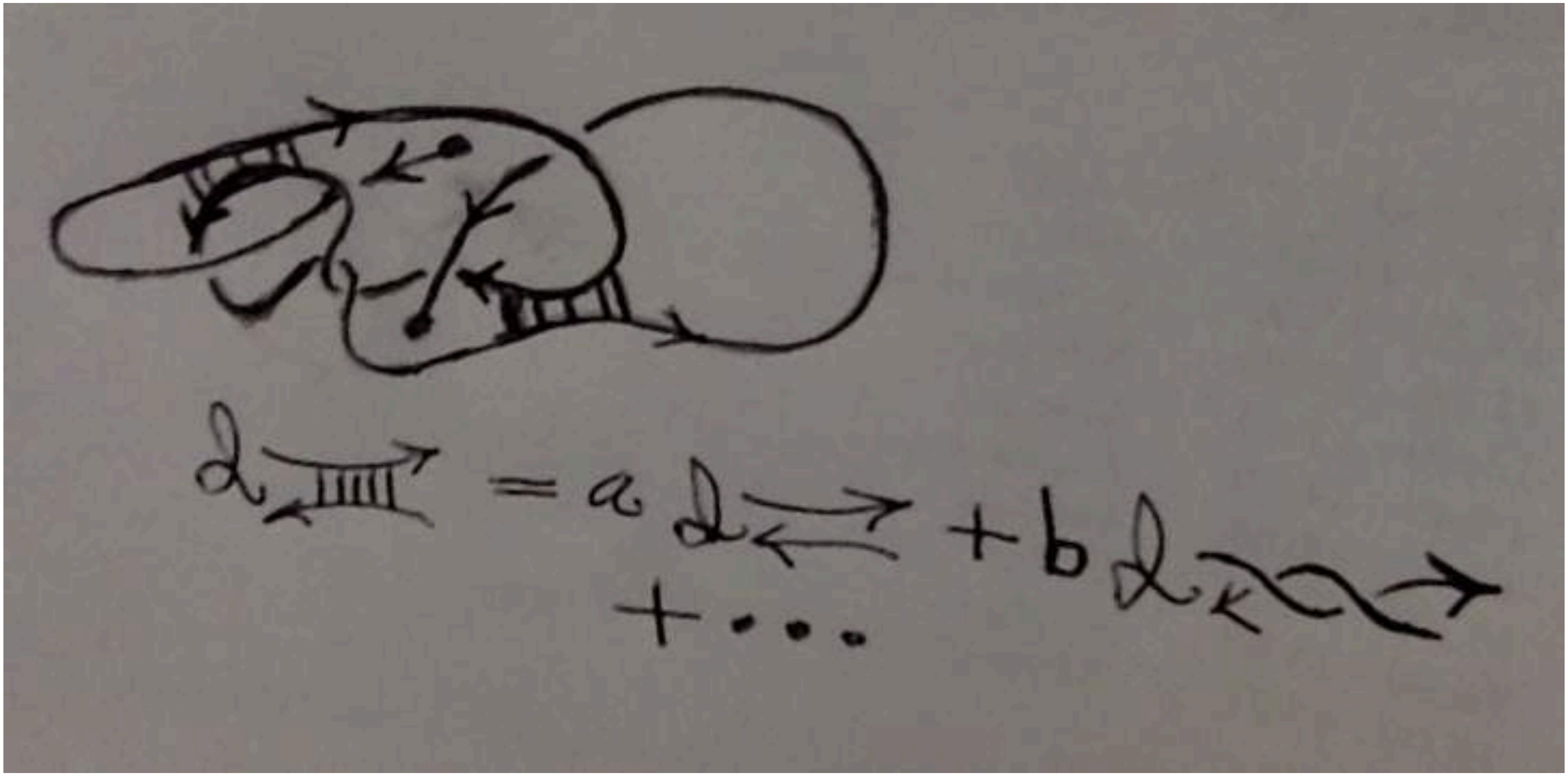


	w_+	w_-
A	0	0
B	1	−1
C	−1	1
D	1	−1
E	−1	1

FIGURE 41. The weight chart of K

Is the height of this knotoid 1 or 2?

Next: Topological Invariants of Folded Protein Knotoids



(Expand folding vertex and evaluate a convenient invariant of knotoids. e.g. knotoid bracket.)

QUESTIONS

(1) Determination of the kernel of the virtual closure map: We have nontrivial virtual knotoids closing virtually to the trivial knot. However nontrivial knot-type knotoids close to nontrivial knots. *Is there a proper knotoid (a classical knotoid with nonzero height) whose virtual closure is the trivial knot?*

(2) Determination of the image of the virtual closure map: We show that the virtual closure map is not a surjective map. The proof will appear in [11]. Here we ask the following question. *How to determine if a given virtual knot is in the image of* $\overline{v}$*?*

(3) A generalization of the first question: *Is there a proper knotoid whose virtual closure is a classical knot* or *do proper knotoids always close (virtually) to a virtual knot of genus 1?*

(4) Conjecture: *The Jones polynomial for knotoids in* S^2 *detects the triviality of classical knotoids.* Let $\overline{K}$ be a virtual knot with trivial Jones polynomial. If the conjecture holds, we will be able to conclude that the virtual closure of any proper knotoid is nontrivial, by using the equality $V(K) = V(\overline{v}(K))$, where V denotes the Jones polynomial.

(5) We want to know more about the height of knotoids and its relations with both the affine index polynomial and the arrow polynomial. We have given examples where the estimation of the arrow polynomial is more powerful than the affine index polynomial in detecting the height of a given classical knotoid. Does there exist an example for which the index polynomial is superior to the arrow polynomial in height determination?

(6) Khovanov homology can be extended to an invariant of knotoids. There is a direct analog of Khovanov homology for classical knotoids. The analogs of Khovanov homology for virtual knots [7, 30] can be applied to virtual knotoids. It is worth investigating Khovanov homology for knotoids. We can ask the following question: *Does Khovanov homology for knotoids detect the trivial knotoid?* Note that Khovanov homology detects the unknot [22].

(7) Let C be an open oriented curve in 3-dimensional space. The set of knotoids associated to C that are obtained by projecting the curve to planes deserves investigation since the physical properties of the curve can be studied in this way.

THANK YOU FOR YOUR ATTENTION!